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Correction of First Mid-term
Math 107 (Summer Semester 28/29)

Question 1 [3]

$$BC = \begin{pmatrix} 3 & 4 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 5 & -1 & 2 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 19 & 1 & 6 \\ 9 & -3 & 4 \end{pmatrix} \quad ①$$

$$2BC - D = \begin{pmatrix} 38 & 2 & 12 \\ 18 & -6 & 8 \end{pmatrix} - \begin{pmatrix} 36 & -2 & 11 \\ 16 & -12 & 7 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 1 \\ 2 & 6 & 1 \end{pmatrix}$$

$$A = 2BC - D \iff \begin{pmatrix} x & 2x & y \\ 2 & 3x & x-y \end{pmatrix} = \begin{pmatrix} 2 & 4 & 1 \\ 2 & 6 & 1 \end{pmatrix} \quad ①$$

then $x = 2$ and $y = 1$. ①

Question 2 [4]

This system can be written as $AX = B$ with $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 2 & (a^2+1) \end{pmatrix}$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \quad ①$$

The system has a unique solution if and only if $\det A \neq 0$

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 2 & (a^2+1) \end{vmatrix}\begin{matrix} 1 & 1 \\ -1 & 1 \\ 0 & 2 \end{matrix} = 2(a^2-1) = 2(a-1)(a+1) \quad ②$$

$$\det A \neq 0 \iff a \in \mathbb{R} - \{\pm 1\}. \quad ①$$

Question 3: [3]

$$\begin{cases} 2x + y - z + w = 4 \\ x + y + w = 2 \end{cases} \iff \begin{pmatrix} 2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 4 \\ 1 & 0 & 1 & 1 & 2 \end{array}\right) = \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & -3 & -1 & 0 \end{array}\right) \begin{matrix} R_2 \\ R_1 - 2R_2 \end{matrix}$$

Then $\begin{cases} x + z + w = 2 \\ y - 3z - w = 0 \end{cases}$. So $\begin{cases} x = 2 - z - w \\ y = 3z + w \\ z \text{ and } w \text{ are free} \end{cases}$ ②

The set of solution is:

$$S = \{(2-z-w;\ 3z+w;\ z;\ w) \,/\, z, w \in \mathbb{R}\}$$
$$= \{(2,0,0,0) + z(-1,3,1,0) + w(-1,1,0,1) \,/\, z, w \in \mathbb{R}\}$$

①

Question 4 [4]

As $\det A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$ then the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$ has not an inverse.

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Question 5 [6]

$$\mathrm{Adj}(A) = \begin{pmatrix} +\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} & +\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \\ +\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} \end{pmatrix}^T$$ 3

$$= \begin{pmatrix} -1 & -7 & 5 \\ -7 & 5 & -1 \\ 5 & -1 & -7 \end{pmatrix}^T = \begin{pmatrix} -1 & -7 & 5 \\ -7 & 5 & -1 \\ 5 & -1 & -7 \end{pmatrix}$$ 1

$\det A = \begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -18 \neq 0$ then A has an inverse.

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$$A^{-1} = \frac{1}{\det A}\,\mathrm{adj}(A)$$ 1

$$A^{-1} = -\frac{1}{18}\begin{pmatrix} -1 & -7 & 5 \\ -7 & 5 & -1 \\ 5 & -1 & -7 \end{pmatrix} = \begin{pmatrix} 1/18 & 7/18 & -5/18 \\ 7/18 & -5/18 & 1/18 \\ -5/18 & 1/18 & 7/18 \end{pmatrix}.$$

Question 6 [4]

$$\begin{vmatrix} 1 & 2 & 3 & 1 \\ -1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 3 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 3 \end{vmatrix} + 2\begin{vmatrix} 2 & 3 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$ 2

$$= 4 - 2 - 16 = -14.$$ 2

Question 7 [6]

As $A^T = \begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix}$ then $A = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$

$$A^2 = A \cdot A = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 8 & 3 \\ -3 & -1 \end{pmatrix} \quad 1$$

$$A^2 - 3A = \begin{pmatrix} 8 & 3 \\ -3 & -1 \end{pmatrix} - 3\begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2 \quad 2$$

$$A^2 - 3A = -I \Leftrightarrow 3A - A^2 = I$$

$$\Leftrightarrow A(3I - A) = I$$

We deduce that A has an inverse and its inverse is:

$$A^{-1} = 3I - A = \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}. \quad 1 \qquad 2$$