Complex Analysis review notes for weeks 1-6

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In what follows, unless stated otherwise a "domain" is a connected open set. Generally we do not include the boundary of the set, although there are many cases where we consider functions which extend continuously to the boundary.

1 Week 1

We will say that a function $f : D \to \mathbb{C}$ defined on a domain D is *analytic* at a point $z_0 \in D$ if the derivative of $f(z)$ exists at z_0, i.e.:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists,}$$

where the limit above is a complex limit; in particular, the limit from every direction must exist and be equal. If D is an open set, we say f is analytic on D if it is analytic at every point in D.

Comparing the limits in the "real direction" and the "imaginary direction" leads to the *Cauchy-Riemann equations*: if $f(z) = u(x, y) + iv(x, y)$, where u and v are real-valued and $z = x + iy$, then f is analytic if and only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

As a consequence of the C-R equations, if f is analytic and if u and v are twice-differentiable functions, then u and v are also *harmonic* functions:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

As we will see later, the assumption that u and v be twice differentiable can be dropped without changing the truth of the statement. If u and v are functions satisfying the Cauchy-Riemann equations than v is called a *harmonic conjugate* of u (and $-u$ is a harmonic conjugate of v).

1.1 Existence of harmonic conjugates

Given a single harmonic function u on a domain D, the question arises: can we find a harmonic conjugate to u, or in other words is u the real part of some analytic function f? The answer depends quite heavily on the nature of the domain D. If D is a disk with centre (x_0, y_0), then v can be constructed explicitly by integration:

$$v(x,y) = C - \int_{x_0}^{x} \frac{\partial u}{\partial y}(s, y_0)\, ds + \int_{y_0}^{y} \frac{\partial u}{\partial x}(x, t)\, dt,$$

where C is an arbitrary constant which determines the value of $v(x_0, y_0)$. This argument can be generalized to some other regions (e.g. rectangles or star-shaped regions), but not to arbitrary regions: the function $u(x,y) = \frac{1}{2}\log(x^2 + y^2)$ has no harmonic conjugate on the punctured plane $\mathbb{C} \setminus \{0\}$. Later in the course we see that the problem with the punctured plane is that it is not simply connected.

1.2 Conformality

Another important feature of analytic functions is conformality. We require a few definitions at this point. A *path* is a differentiable function $\gamma : I \to \mathbb{C}$, where I is a sub-interval of the real line. The *tangent vector* to a path $\gamma(t)$ is just the derivative $\gamma'(t) \in \mathbb{C}$, considered as a vector in the plane. A function f is *conformal* at a point z_0 if and only if for any two paths γ and ζ which pass through z_0 the angle between the tangent vectors to γ and ζ at z_0 is equal to the angle between the tangent vectors to $f \circ \gamma$ and $f \circ \zeta$ at $f(z_0)$.

Suppose f is analytic at z_0 and $f'(z_0) \neq 0$. Then by the chain rule if $z_0 = \gamma(t_0)$, then $(f \circ \gamma)'(z_0) = f'(z_0)\gamma'(t_0)$; in other words the effect of f on $\gamma'(t_0)$ is to scale it by $|f'(z_0)|$ and rotate it by $\arg(f'(z_0))$. Hence the tangent vector to any path through z_0 will be rotated by the same amount; hence f is conformal at z_0. Similarly if f is conformal at z_0, then f is also analytic at z_0 and has non-zero derivative, although the proof of this is slightly more complicated. (It's in Ahlfors if you're interested.)

Hence *analytic functions preserve angles*, except at critical points. In fact, if z_0 is not a critical point of f then in some neighbourhood of z_0 the analytic function f acts approximately just like multiplication by $f'(z_0)$. This hints at many other properties of analytic functions to come, such as the fact that local inverses to analytic functions exist (away from critical points, of course) and are analytic or the fact that maxima of $\Re(f)$ or $|f|$ do not occur in the interior of the domain of f if f is non-constant.

1.3 Fractional Linear Transformations

Finally, as a specific example of conformal maps consider the fractional linear transformations. A *fractional linear transformation* or *FLT* is just a function of the form

$$f(z) = \frac{az+b}{cz+d},$$

where a, b, c, $d \in \mathbb{C}$ are constants such that $ad-bc \neq 0$. We can extend such a function f to the *extended complex plane* or *Riemann sphere* $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ by defining $f(\infty) = a/c$ if $c \neq 0$ and ∞ otherwise, and by defining $f(-d/c) = \infty$ if $c \neq 0$.

Such a function f has no critical points and hence is conformal everywhere (except $-d/c$ if $c \neq 0$). FLT's have other interesting properties. In particular, if we extend the definition of "circles" in the Riemann sphere to include lines, which we consider to be "circles passing through ∞", then FLT's take circles to circles. More significantly for this course, we will see later than any bijective conformal map from the unit disk to itself must be an FLT; combined with the Riemann Mapping Theorem, this allows us to classify the set of conformal self-maps of any simply connected open subset of the complex plane.

2 Week 2

Recall that if $f : D \to \mathbb{C}$ is a function and $\gamma : [a, b] \to \mathbb{C}$ is a path whose image is contained in D, then the *path integral* of f along γ is defined to be:

$$\int_\gamma f\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt.$$

If γ is a loop and f is analytic, it doesn't take many examples before one begins to notice a pattern: the path integral of an analytic function around a loop is related to the poles of the function, if any, inside the loop. This simplest case of this is *Cauchy's Theorem*: if f is analytic on a domain D and extends continuously to ∂D, then $\oint_{\partial D} f\,dz = 0$, if ∂D is oriented appropriately (see below).

It's important to note that Cauchy's theorem applies even if the boundary of D is not a single loop: if ∂D has more than one component, just add up the path integrals along each component. However, each component must be oriented correctly. Specifically, each component must be oriented in such a way that as you walk around the resulting path, the interior of D is to your left. E.g. if D is a disk the boundary must be oriented counter-clockwise; if D is an annulus the outer boundary must be oriented counter-clockwise and the inner boundary must be oriented clockwise, and so on.

2.1 Cauchy's and Green's Theorems

There's an interesting argument for Cauchy's Theorem in "Visual Complex Analysis", however the proof we used in class comes from Gamelin's book. Specifically, Cauchy's theorem is a special case of *Green's Theorem*: if $P(x, y)$ and $Q(x, y)$ are real-valued functions differentiable function on the same domain D, then

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dxdy.$$

If $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$, then we can let $P = u + iv$, $Q = -v + iu$, and $dz = dx + idy$. Then the left-hand side of Green's Theorem above is equal to $\int_{\partial D} f\,dz$, while on the right-hand side we get 0 since by the Cauchy-Riemann equations $\partial Q/\partial x = \partial P/\partial y$.

Green's theorem can in turn be proved very directly, if you don't mind all the bookkeeping. First, prove Green's Theorem directly in the case where D is a triangle with two sides parallel to the coordinate axes, by directly simplifying each side of the equation. Second, prove Green's Theorem in the case where D is the image of such a triangle under a continuous bijective map, using the appropriate change of variables on each side. (That's the messy part.) Finally, prove the theorem in the case where D is the union of such images, by noting that where two such images share a "side" the corresponding line integrals cancel out. This establishes Green's Theorem, and hence Cauchy's, for any domain in the scope of this course.

2.2 Existence of antiderivatives

Note that Green's Theorem is one of the multivariable generalizations of the Fundamental Theorem of Calculus. This suggests that there may be a "Fundamental Theorem of Calculus" for complex path integrals too, and there is, sort of. If $f(z)$ has an anti-derivative $F(z)$ on D, also called a *primitive*, then $\int_\gamma f(z)\,dz = F(b) - F(a)$ for all paths γ in D from a to b. However, unlike in the real case $f(z)$ may not have such a primitive even if f is analytic; it depends both on f and on the shape of D. For example, $f(z) = 1/z$ has no primitive on the punctured plane $\mathbb{C} \setminus \{0\}$, but does have a primitive on the slit plane $\mathbb{C} \setminus \{x \in \mathbb{R}, x \leq 0\}$. Once again, the notion of "simply connected" rears its head.

2.3 Cauchy's Integral Formula

To move from Cauchy's Theorem to Cauchy's Integral formula, we first note the *Mean Value Property* for harmonic functions. If $u(x, y)$ is a harmonic function defined on a disk of radius r centred at z_0, then

$$\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta = u(z_0).$$

In other words, the average value of u on the boundary of the disk is equal to the value of u at the centre. This is an important property of harmonic functions, and a consequence of Green's Theorem. Now suppose that $f(z)$ is analytic in a neighbourhood of z_0 and that γ is the boundary of the disk $\{|z-z_0| \leq r\}$, oriented counter-clockwise. Then by substituting $z = z_0+re^{i\theta}$, taking real and imaginary parts, and applying the Mean Value Property, we get

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\,dz = f(z_0).$$

Now suppose that f is analytic on a domain D and extends continously to ∂D, and fix a point z_0 in the interior of D. Let D_0 be an open disk of radius r around z_0, where $r > 0$ is chosen small enough that $D_0 \subset D$, and let $D_1 \subset D$ be the complement of the closure of D_0. Since $z_0 \notin D_1$, $f(z)/(z - z_0)$ is analytic in D_1, and by Cauchy's Theorem

$$\frac{1}{2\pi i}\int_{\partial D_1} \frac{f(z)}{z - z_0}\,dz = 0.$$

But if we orient the boundaries correctly (as described above re: Cauchy's Theorem), $\int_{\partial D} g\,dz = \int_{\partial D_1} g\,dz + \int_{\partial D_0} g\,dz$ for any integrable function g. Hence we arrive at *Cauchy's Integral Formula*:

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f(z)}{z-z_0}\,dz = f(z_0).$$

2.4 Analytic functions are "C^∞"

If we substitute the left-hand side of the above formula into the definition of $f'(z)$, we get the following variant of Cauchy's Integral Formula for any integer $m \geq 0$:

$$\frac{m!}{2\pi i}\int_{\partial D}\frac{f(z)}{(z-z_0)^{m+1}}\,dz = f^{(m)}(z_0).$$

Almost as a side-effect, we get the important fact that an analytic function has derivatives of all orders. This is one of the most striking differences between complex and real analysis: there is no such thing as a complex function which can only be differentiated a finite number of times.

3 Week 3

3.1 Maximum Principle

We start with a consequence of the Mean Value Property: the *Maximum Principle* for harmonic functions. Namely, if u is a harmonic function on a domain D and if u achieves a global maximum in the interior of D, then u is a constant function.

The proof of the Maximum Principle is an example of a classic topological argument. Let M be the global maximum value of u, let $A = \{(x,y) \in D | u(x,y) = M\}$, and let $B = \{(x,y) \in D | u(x,y) < M\}$. We will show that both A and B are open subsets of D. Since $D = A \cup B$, A and B are disjoint, and D is connected, this proves that one of A or B must equal D and the other must be empty. By assumption A is non-empty, so we must have $A = D$ and hence u is constant.

B is an open subset of D since u is continuous. To show A is an open subset, suppose $(x_0, y_0) \in A$ and pick $t > 0$ such that the disk around (x_0, y_0) of radius t is contained in D. By the Mean Value Property, for any $r < t$ the average value of u on the circle of radius r centred at (x_0, y_0) must be equal to $u(x_0, y_0)$, which is M. But since M is a global maximum value, the only way this is possible is if $u(x, y) = M$ for all (x, y) on the circle of radius r, and this is true for all $r < t$. Hence the open disk $\{||(x, y) - (x_0, y_0)|| < t\}$ is contained in A, and hence A is an open subset of D, completing the proof.

If we make the further assumptions that D is a bounded domain and that u extends continuously to ∂D, then the Maximum Principle implies that u must achieve its maximum value on ∂D. For by compactness u must achieve its maximum value somewhere on $D \cup \partial D$; either u is non-constant, in which case by the Maximum Principle the maximum value is not achieved in the interior, or else u is constant on D, in which case u is constant on $D \cup \partial D$ by continuity. In either case the maximum value of u is achieved on the boundary.

Obviously there is a corresponding Minimum Principle for harmonic functions as well.

3.2 Maximum Modulus Principle

Clearly we can apply the Maximum Principle to analytic functions and say that for an analytic function f, if the maximum (or minimum) value of $\Re(f)$ (or $\Im(f)$) is achieved in the interior of the domain, then f is a constant function. (There is one extra step–we need to show that if $\Re(f)$ is constant then so is $\Im(f)$ and vice-versa. This is almost, but not quite, trivial. The Cauchy-Riemann equations prove that the first derivatives of $\Im(f)$ are zero everywhere if $\Re(f)$ is constant. Then apply the Mean Value Theorem from first-year calculus.)

But we can also derive the *Maximum Modulus Principle* for analytic functions. Suppose $|f(z)|$ achieves its maximum value at a point z_0 in the interior of the domain D. If $|f(z_0)| = 0$ is the maximum value then clearly f is constant, so suppose $f(z_0) \neq 0$. Let $\lambda \in \mathbb{C}$ be the unique number such that $|\lambda| = 1$ and such that $\lambda f(z_0)$ is both real and positive, and let $g(z) = \lambda f(z)$. Then $\Re(g(z_0)) = |g(z_0)|$, while for all $z \in D$ we have $\Re(g(z)) \leq |g(z)| = |f(z)| \leq |f(z_0)| = |g(z_0)|$. So $\Re(g)$ achieves its global maximum at z_0, and hence $\Re(g(z)) = \Re(g(z_0)) = |g(z_0)|$ for all z. Since $|g(z_0)| = |f(z_0)|$ is a global maximum, we must also have $\Im(g(z)) = 0$ for all z. Hence $g(z)$ and $f(z)$ are constant functions of z. In other words, if $|f(z)|$ achieves its maximum value in the interior of its domain then $f(z)$ is a constant function.

And again, if we further assume that D is a bounded domain and that f

extends continuously to ∂D, then we can conclude that $|f(z)|$ must achieve its global maximum value somewhere on ∂D. Note that the assumption that f extends continuously to ∂D is essential; consider $f(z) = 1/z$ on the disk $D = \{|z-1| < 1\}$.

3.3 Schwarz's Lemma

The first consequence of the Maximum Modulus Principle is *Schwarz's Lemma*. Let $D = \{|z| < 1\}$ and suppose f is analytic on D. Suppose further that f maps D into itself, i.e. $|f(z)| \leq 1$ for all $z \in D$, and suppose $f(0) = 0$. Schwarz's Lemma states that either $|f(z)| < |z|$ for all $z \in D$ except $z = 0$, or else there exists a complex number λ such that $f(z) = \lambda z$ for all $z \in D$ and $|\lambda| = 1$.

In other words, any analytic map from the open unit disk to itself which fixes the origin is either a contraction (i.e. $|f(z)| < |z|$) or a rotation. Note that in either case we have $|f(z)| \leq |z|$ for all z.

To prove Schwarz's Lemma, we define a new analytic function $F(z)$ on D as follows:

$$F(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases}$$

Note that if $|z| = r$, where $r < 1$, then $|F(z)| = |f(z)|/r \leq 1/r$. By the Maximum Modulus Principle applied to $\{|z| \leq r\}$, we see that $|F(z)| \leq 1/r$ for all $|z| \leq r$. Letting r approach 1, we get $|F(z)| \leq 1$ for all $z \in D$, i.e. $|f(z)| \leq |z|$ for all $z \in D$. Suppose that $|f(z_0)| = |z_0|$ for some $z_0 \in D$, $z_0 \neq 0$. Then $|F(z_0)| = 1$, and by the Maximum Modulus Principle F is a constant function on the disk $\{|z| \leq r\}$ for all r such that $|z_0| < r < 1$. Hence F must be constant on the open disk $\{|z| < 1\}$, i.e. $F(z) = \lambda$, and $|\lambda| = |F(z_0)| = 1$. Hence $f(z) = \lambda z$ for all $z \in D$, completing the proof.

3.4 Liouville's Theorem

We can of course generalize Schwarz's Lemma to disks with other radii and centred at other points. Suppoze f is an analytic function on the disk $D = \{|z-a| < N\}$ and suppose $f(a) = b$ and $|f(z)-b| \leq M$ whenever $|z-a| < N$. Then we can define an analytic function $F(z)$ as follows:

$$F(z) = \begin{cases} \frac{f(z)-b}{z-a}, & z \neq a \\ f'(a), & z = a \end{cases}$$

Then arguing as before, we can conclude that $|F(z)| \leq M/N$ for all $z \in D$. But now suppose we make the further assumption that $f(z)$ is analytic everywhere on $\mathbb{C}$, and $|f(z)-b| \leq M$ for all $z \in \mathbb{C}$. Then we can let $N \to \infty$, and conclude that $|F(z)| \leq 0$ for all $z \in \mathbb{C}$–or in other words, $f(z) = b$ for all z. This is *Liouville's Theorem*: bounded entire functions are constant. (Recall that an *entire* function is a function which is analytic everywhere on $\mathbb{C}$.)

3.5 Conformal self-maps of the disk

One significant application of Schwarz's Lemma is the classification of all conformal self-maps of the unit disk. Suppose $f : D \to D$ is a bijective analytic function, where $D = \{|z| < 1\}$. What are the possible choices for f?

If $f(0) = 0$, then Schwarz's Lemma implies that $|f(z)| \leq |z|$ for all $z \in D$. At the same time, f^{-1} is also conformal and hence analytic; so $|f^{-1}(z)| \leq |z|$ for all $z \in D$, or in other words (since f is a bijection) $|z| \leq |f(z)|$ for all z. Hence $|f(z)| = |z|$ for all $z \in D$, and by Schwarz's Lemma f must be a rotation, i.e. $f(z) = e^{i\theta} z$ for some θ.

Suppose $f(0) \neq 0$. Then $f(a) = 0$ for some $a \neq 0$ since f is a bijection. Let g be the following fractional linear transformation:

$$g(z) = \frac{z-a}{1-\bar{a}z}.$$

By direct calculation we can show that $|g(e^{i\phi})| = 1$ for all ϕ, hence $|g(z)| \leq |z|$ for all $z \in D$ by the Maximum Modulus Principle. Thus, g is also a bijective analytic map from D to D, and $g(a) = 0$. Hence $(f \circ g^{-1})(0) = 0$, so $f \circ g^{-1}$ is a rotation by the previous case. Hence there exists θ such that

$$\begin{aligned} f(z) &= (f \circ g^{-1})(g(z)) \\ &= e^{i\theta} g(z) \\ &= e^{i\theta} \frac{z-a}{1-\bar{a}z}. \end{aligned}$$

We have proved that *every bijective conformal self-map of the open unit disk has the above form* for some a and θ. In particular, every conformal self-map of the open unit disk is a fractional linear transformation.

This result, combined with the Riemann Mapping Theorem, allows us to completely determine the set of possible conformal self-maps of any simply connected proper subset B of the complex plane. Simply find a single bijective analytic map h from B to the open unit disk, and then every conformal self-map of B is of the form $h^{-1} \circ f \circ h$, where f is a self-map of the open unit disk as described above.

3.6 Pick's Lemma

We have one final result to cover for this section. Suppose f is an analytic function on the open unit disk with $|f(z)| < 1$ for all $|z| < 1$. If $f(0) = 0$, then Schwarz's Lemma implies that $|f'(0)| \leq 1$. More generally, if $f(z) = w$, then we can find conformal self-maps of the unit disk, g and h, such that $g(0) = z$ and $h(w) = 0$. Then $h \circ f \circ g$ is an analytic function from the unit disk to itself, and $(h \circ f \circ g)(0) = 0$. By Schwarz's Lemma again, we get $|(h \circ f \circ g)'(0)| \leq 1$, and hence by the chain rule $|h'(w)f'(z)g'(0)| \leq 1$. But we can calculate $|g'(0)|$ and $|h'(w)|$ directly; when we do we get *Pick's Lemma*:

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

for all $|z| < 1$. Furthermore, equality holds if and only if f is also a conformal self-map of the unit disk.

Pick's Lemma is in a sense an analogue to Schwarz's Lemma for hyperbolic geometry. The right-hand side of the inequality in Pick's Lemma is related to the hyperbolic metric on the open unit disk. Thus where Schwarz's Lemma says that an analytic function from the unit disk to itself is either a contraction or a rotation, Pick's Lemma says that an analytic function from the unit disk to itself either shrinks hyperbolic distance, or else is a conformal self-map of the unit disk. Conformal self-maps of the unit disk *are* the isometries of the hyperbolic plane.

4 Weeks 4-5

(I don't have separate notes for these two weeks; it seems my lecture plan for week four bled over into the fifth week. As a consequence, this section is particularly long.)

4.1 Review of singularities and residues

Recall that an analytic function f which is defined on a punctured disk $\{0 < |z - z_0| < \epsilon\}$ is said to have a *singularity* at z_0, and that singularity is either a *removable singularity*, a *pole*, or an *essential singularity*, depending on the behaviour of $f(z)$ in the limit as z approaches z_0. If $f(z)$ approaches a finite limit, then the singularity is removable, e.g. $(z^2 - 1)/(z - 1)$ as z approaches 1. If $f(z)$ approaches infinity, then the singularity is a pole, e.g. $1/z$ as z approaches 0. Finally, if $f(z)$ has no limit as z approaches z_0, then the singularity is essential, e.g. $e^{1/z}$ as z approaches 0. In the case of an essential singularity, f will assume every possible finite value infinitely often as z approaches the singularity. A function which is analytic except for some poles is called *meromorphic*.

Also recall that the *residue* of an analytic function f at a point z_0, where z_0 is either a point in its domain or a singularity, is defined roughly as the coefficient of $1/(z - z_0)$ in the Laurent expansion of f, or more formally as

$$Res(f, z_0) = \frac{1}{2\pi i} \oint_{|z-z_0|=\epsilon} f(z)\, dz.$$

If f is defined everywhere in a domain D except at a finite number of singularities, then by the *Residue Theorem* we have:

$$\int_{\partial D} f(z)\, dz = 2\pi i \sum_{j=1}^{m} Res(f(z), z_j),$$

where z_1, z_2, $\ldots$, z_m are the singularities. The Residue Theorem can be proven using Cauchy's Theorem, to equate the integral on the left with the sum of the integrals of f over small loops around each of the singularities.

4.2 The Argument Principle

Now suppose that $f(z)$ is a meromorphic function. If z_0 is either a zero or a pole of f, then in some neighbourhood of z_0 we have $f(z) = (z - z_0)^m g(z)$, where $m \neq 0$ is an integer and g is an analytic (not meromorphic) function with $g(z_0) \neq 0$. Then by direct calculation,

$$\frac{f'(z)}{f(z)} = \frac{m}{z - z_0} + \frac{g'(z)}{g(z)}.$$

The second term on the right-hand side is analytic at z_0, hence the residue of $f'(z)/f(z)$ at z_0 is exactly m. Thus applying the Residue Theorem to $f'(z)/f(z)$ gives the *Argument Principle* for meromorphic functions:

$$\int_{\partial D} \frac{f'(z)}{f(z)}\, dz = 2\pi i(N_0 - N_\infty),$$

where N_0 is the number of zeroes of f in D, counting multiplicity, and N_∞ is the number of poles of f in D, again counting multiplicity.

4.3 Change in argument over a curve

We pause here to provide an alternate interpretation of the integral on the left-hand side. In real analysis, we would no doubt integrate $f'(z)/f(z)$ by noting that it's the derivative of $\log(f(z))$. In complex analysis, log is not a well-defined function. However, the differential $d\log(z)$ is still well-defined. Thus, if γ is a loop and if f has no zeroes or poles on γ, we may write:

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz &= \frac{1}{2\pi i}\int_\gamma d\log f(z) \\ &= \frac{1}{2\pi i}\int_\gamma d\log|f(z)| + \frac{1}{2\pi}\int_\gamma d\arg f(z).\end{aligned}$$

Now $\log|f(z)|$, unlike $\log f(z)$ or $\arg f(z)$, is a well-defined function. (In other words $d\log|f(z)|$ is an *exact* differential form.) Hence the left-hand integral in this last expression will equal zero, and thus

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi}\int_\gamma d\arg f(z).$$

Thus even though $\arg f(z)$ is not a well-defined function, we can still say that the integral of $f'(z)/f(z)$ along γ is measuring the *change in argument* of f along γ. In particular, the quantity $N_0 - N_\infty$ is equal to the change in argument of $f(z)$ around ∂D divided by 2π.

4.4 Rouché's Theorem

This will lead us to the topic of winding numbers, but first we will apply this concept to *Rouché's Theorem*. Suppose that f and h are analytic functions on a bounded domain D which extend continuously to ∂D, and suppose that $|h(z)| < |f(z)|$ for all $z \in \partial D$. In particular, $f(z) \neq 0$ and $h(z) + f(z) \neq 0$ for all $z \in \partial D$. Then since argument behaves like logarithm with respect to products, we have the following:

$$d \arg(f(z) + h(z)) = d \arg(f(z)) + d \arg \left(1 + \frac{h(z)}{f(z)}\right).$$

Since $|h(z)| < |f(z)|$ for all $z \in \partial D$, the image of ∂D under the map $z \mapsto 1 + \frac{h(z)}{f(z)}$ is entirely contained in the right half of the complex plane, which is a star-shaped region which doesn't contain the origin. Therefore the total change in argument of $1 + \frac{h(z)}{f(z)}$ along ∂D must equal zero, and hence the total change in argument of $f(z) + h(z)$ along ∂D equals the total change of argument of $f(z)$ along ∂D. Therefore f and $f + h$ have the same number of roots in D; this is Rouché's Theorem.

Perhaps the most well-known application of Rouché's Theorem is to prove the Fundamental Theorem of Algebra, that every polynomial with complex coefficients has a root. Suppose $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial, where a_0, a_1, $\ldots$, a_{n-1} are complex constants. Let $f(z) = z^n$ and $h(z) = p(z) - f(z)$. By choosing a constant M to be sufficiently large (anything larger than n times the maximum absolute value of the a_i's does the trick), we can ensure that $|f(z)| > |h(z)|$ for all $|z| = M$, and hence by Rouché's Theorem p and f have the same number of zeroes in $\{|z| < M\}$ counting multiplicity.

4.5 Winding numbers

Now we come to *winding numbers*. Given a path γ and a point z_0 which does not lie in the image of γ, we define the winding number of γ around z_0 as follows:

$$W(\gamma, z_0) = \frac{1}{2\pi} \int_\gamma d \arg |z - z_0| = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0}\, dz.$$

Note that γ does not need to be a loop, but the winding number is only guaranteed to be an integer if γ is a loop. Now suppose that γ is a loop and that f is a meromorphic function defined on a domain for which γ is the boundary. Then we have

$$\begin{aligned} W(f \circ \gamma, 0) &= \frac{1}{2\pi i} \int_{f \circ \gamma} \frac{1}{z}\, dz \\ &= \frac{1}{2\pi i} \int_{\gamma} \frac{f'(w)}{f(w)}\, dw. \end{aligned}$$

where the last step used the substitution $z = f(w)$. Hence $W(f \circ \gamma, 0) = N_0 - N_\infty$.

More generally, if f is an analytic function, then $W(f \circ \gamma, w_0)$ measures the number of points $z \in D$ such that $f(z) = w_0$, counting multiplicity. In particular, if w_0 lies outside the image of ∂D under f (as measured by the winding number) then w_0 does not lie in the image of D under f. This is another property of analytic functions, related to the Maximum Modulus Principle, that distinguishes analytic functions from continuous two-variable real functions in the plane.

4.6 Generalization to non-analytic functions

Note that even if f is a continuous non-analytic function, the winding number $W(f \circ \gamma, w_0)$ may still be defined, and with it we can attempt to define a "topological argument principle". Specifically, suppose $f : D \to \mathbb{C}$ is a continuous function of two variables which extends continuously to ∂D for which $f(z) \neq w_0$ for any $z \in \partial D$, and suppose ∂D is the image of a path γ, appropriately oriented. Then hopefully $W(f \circ \gamma, w_0)$ will equal the number of points $z \in D$ for which $f(z) = w_0$, "counting multiplicity". The key here is to correctly define what we mean by the "multiplicity" of a point for a non-analytic function. For a continuous two-variable function f, the "multiplicity" of a point z in the domain of f is best defined as the *degree* of f at that point.

If f is differentiable at z (in the real, two-variable sense) then the degree can sometimes in turn be determined from the Jacobian of f: the degree is $+1$ if the determinant of the Jacobian is positive, -1 if the determinant of the Jacobian is negative. If f is an arbitrary continuous function (or if the determinant of the Jacobian is zero) then the degree of f at z can be defined as $W(f \circ \rho, f(z))$, where ρ is a counter-clockwise loop around z, small enough not to contain any other points w such that $f(z) = f(w)$. This may seem somewhat tautological...claiming that winding number measures the number of points counting multiplicity, and then defining multiplicity in terms of the winding number...but remember that we don't have Cauchy's theorem for non-analytic functions, so the fact that windings numbers around ∂D have anything to do with winding numbers around these smaller curves is significant.

A key difference between analytic and non-analytic functions in this context is that if f is analytic, the multiplicity of a point z with respect to f is always a positive integer, corresponding to the degree of the first non-zero non-constant term in the Taylor series around that point. For non-analytic functions, the multiplicity of a point may be zero or negative. For example the multiplicity of $z = 0$ with respect to the function $f(x + iy) = x + i|y|$ is 0, while the multiplicity of $z = i$ is $+1$ and the multiplicity of $z = -i$ is -1. Nevertheless the "topological argument principle" applies: if $\gamma(t) = 2e^{it}$ for $0 \leq t \leq 2\pi$, then $W(f \circ \gamma, i) = 0$, which is the sum of the multiplicities of the two points which are mapped to i by f.

It should be stressed that the "topological argument principle" is not a hard and fast theorem, and can fail if the function f is sufficiently complicated. Think of it an a useful analogy more than anything else.

4.7 Brouwer's Fixed Point Theorem

Let g be a continuous map from the closed unit disk $\{|z| \leq 1\}$ to itself; *Brouwer's Fixed Point Theorem* states that g must have a fixed point, i.e. there exists a point z_0 such that $g(z_0) = z_0$. This is one of the most famous results in topology, and can be proved using winding numbers.

Assume instead that $g(z) \neq z$ for all z; we will find a contradiction. Let $f(z) = -z$, and let $m(z) = g(z) + f(z)$. Let $\gamma(t) = e^{it}$ be the usual path around the unit circle. Since the image of g is contained inside the closed unit disk, $|g(\gamma(t))| \leq 1 = |f(\gamma(t))|$ for all t. Hence the image of γ under the function $z \mapsto 1 + g(z)/f(z)$ must be contained inside the closed disk of radius 1 centred at 1. The image also does not pass through 0, since if $1 + g(z)/f(z) = 0$ we would have $g(z) = -f(z) = z$ which by assumption we do not. Hence just as in the proof of Rouché's Theorem the image of γ under this function is contained in a star-shaped region which does not contain the origin, namely the set $\{|z-1| \leq 1\} \setminus \{0\}$, and hence the change in argument of $1 + g(z)/f(z)$ around γ is zero. Since $m(z) = f(z)(1 + g(z)/f(z))$ we have $W(m \circ \gamma, 0) = W(f \circ \gamma, 0) = 1$.

But now let $\gamma_r(t) = re^{it}$ be the circle of radius r. So γ_0 is a constant path and clearly $W(m \circ \gamma_0, 0) = 0$, while $\gamma_1 = \gamma$ and hence $W(m \circ \gamma_1, 0) = 1$. And by assumption there is no value of r for which the image of γ_r passes through 0, so $m \circ \gamma_r$ never crosses the origin as r varies from 0 to 1, and yet $W(m \circ \gamma_0) \neq W(m \circ \gamma_1)$. This is the desired contradiction.

4.8 The Jump Theorem and the Jordan Curve Theorem

A *simple closed curve* is just the image of a path which forms a loop but doesn't otherwise intersect itself; in other words $\Gamma \subset \mathbb{C}$ is a simple closed curve if Γ is the image of a continuous function $\gamma : [a, b] \to \mathbb{C}$ where $[a, b] \subset \mathbb{R}$ is a closed interval, $\gamma(a) = \gamma(b)$, and $\gamma(s) \neq \gamma(t)$ for any other s and t. The *Jordan Curve Theorem* says that *every simply closed curve in* $\mathbb{C}$ *divides* $\mathbb{C}$ *into two regions, an unbounded region and a bounded one.*

While intuitively obvious the Jordan Curve Theorem is very difficult to prove, largely because Γ could be a very complicated curve. In the special case where Γ is the image of a smooth path γ, the Jordan Curve Theorem can be proved winding numbers. Specifically, consider the function $\zeta \mapsto W(\gamma, \zeta)$, defined for all $\zeta \in \mathbb{C} \setminus \Gamma$. We claim that this function only ever takes on two values, either 0 and 1 or 0 and -1, and that ζ is in the unbounded component of $\mathbb{C} \setminus \Gamma$ if $W(\gamma, \zeta) = 0$ and in the unique bounded component if $W(\gamma, \zeta) = \pm 1$.

To begin with, remember that

$$W(\gamma, \zeta) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - \zeta} \, dz.$$

From this expression we can see that $W(\gamma, \zeta)$ is an analytic, hence continuous, function of ζ on $\mathbb{C} \setminus \Gamma$. But since γ is a loop the winding number is always an integer, and a continuous function whose image is in the integers must be *locally constant*. In other words $W(\gamma, \zeta)$ must be constant on each component of $\mathbb{C} \setminus \Gamma$. Furthermore since Γ is a bounded set $W(\gamma, \zeta)$ must equal 0 if ζ is close to ∞, so $W(\gamma, \zeta) = 0$ everywhere in any unbounded component of $\mathbb{C} \setminus \Gamma$.

Now we need to show that $W(\gamma, \zeta)$ actually changes as you pass from one side of Γ to the other. Specifically, for $z_0 \in \Gamma$ suppose U is a small open disk neighbourhood of z_0 such that Γ divides U into exactly two pieces. Call the piece of the left side (with respect to the direction of γ) U_- and call the piece on the right side U_+. Let γ_0 be the segment of γ lying inside U; γ_0 passes from z_1 on the boundary of U, through z_0, to z_2 on the far side of U. Let γ_- be the other path from z_1 to z_2 on the boundary of U_-, and let γ_+ be the other path from z_1 to z_2 on the boundary of U_+.

Now we have

$$W(\gamma,\zeta) = \frac{1}{2\pi i}\int_{\gamma_0} \frac{1}{z-\zeta}\,dz + \frac{1}{2\pi i}\int_{\gamma\setminus\gamma_0} \frac{1}{z-\zeta}\,dz.$$

Call the first term on the right-hand side $F_0(\zeta)$ and the second term $G(\zeta)$. Note that the function G is actually analytic on the whole of U; it's only F that fails to be defined on γ_0. Now define

$$F_+(\zeta) = \frac{1}{2\pi i}\int_{\gamma_-} \frac{1}{z-\zeta}\,dz + \frac{1}{2\pi i}\int_{\gamma\setminus\gamma_0} \frac{1}{z-\zeta}\,dz,$$

and

$$F_-(\zeta) = \frac{1}{2\pi i}\int_{\gamma_+} \frac{1}{z-\zeta}\,dz + \frac{1}{2\pi i}\int_{\gamma\setminus\gamma_0} \frac{1}{z-\zeta}\,dz.$$

F_+ and F_- are analytic functions that are each analytic on U. Moreover, by Cauchy's Theorem, $F_+(\zeta) = W(\gamma,\zeta)$ for all $\zeta \in U_+$, and similarly $F_-(\zeta) = W(\gamma,\zeta)$ for all $\zeta \in U_-$. Thus the value of $W(\gamma,\zeta)$ jumps from $F_-(\zeta)$ to $F_+(\zeta)$ as ζ crosses from U_- to U_+. The size of the jump can readily be determined:

$$F_-(\zeta) - F_+(\zeta) = \frac{1}{2\pi i}\oint_{\partial U} \frac{1}{z-\zeta}\,dz.$$

But since U is just a disk, the winding number of ∂U around ζ must be ± 1. Hence $W(\gamma,\zeta)$ either increases or decreases by 1 as ζ crosses over Γ.

This is an example of a *Jump Theorem*. More generally, if $f(z)$ is any analytic function we can define a new function F by:

$$F(\zeta) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{z-\zeta}\,dz.$$

Then $F(\zeta)$ will be analytic on $\mathbb{C}\setminus\Gamma$, and by a similar argument to the one above we can show that as ζ crosses Γ the value of $F(\zeta)$ will jump by $\pm f(\zeta)$, depending on the direction of the crossing.

Returning to the Jordan Curve Theorem: since $W(\gamma,\zeta)$ is constant on each component of $\mathbb{C}\setminus\Gamma$, and since $W(\gamma,\zeta)$ changes as ζ crosses Γ, this shows that $\mathbb{C} \setminus \Gamma$ has *at least* two components. It remains to show that $\mathbb{C} \setminus \Gamma$ has *exactly* two components. But since Γ is a compact set, we can cover Γ with a finite number of disks $U_1, U_2, \ldots, U_n$ like the disk U described above. Each U_i is divided by Γ into two subsets which lie in two different components of $\mathbb{C} \setminus \Gamma$ by the above argument. By considering the regions where U_i intersects its neighbours, we can see that the two components of $\mathbb{C} \setminus \Gamma$ corresponding to U_i must be the same as the two components corresponding to its neighbouring U_j's. Hence it's the same two components all the way around, i.e. there are exactly two components of $\mathbb{C} \setminus \Gamma$ which intersect any of the U_i's. But each component of $\mathbb{C} \setminus \Gamma$ must have Γ in its boundary; hence each such component must intersect one of the U_i's. Hence $\mathbb{C} \setminus \Gamma$ has exactly two components. One of those components must be unbounded and hence $W(\gamma, \zeta) = 0$ there; on the other components we must have $W(\gamma, \zeta) = \pm 1$, depending on the orientation of γ. This completes the proof of the Jordan Curve Theorem for smooth curves.

5 An aside on simple connectivity

I honestly didn't know where else to put this; it came up in several different places in the course. So, it gets its own mini-section.

The concept of a *simply connected domain* comes up at several different points in the course. The formal definition is as follows: a subset $D \in \mathbb{C}$ is *simply connected* if every loop γ whose image is in contained in D can be continuously deformed to a point in D. That is, for every loop $\gamma : [a, b] \to D$ there exists a continuous function $G : [0, 1] \times [a, b] \to D$, such that

- for all s, the path $\gamma_s(t) = G(s, t)$ is a loop,
- $\gamma_0 = \gamma$, and
- γ_1 is a constant path, i.e. there exists z_0 such that $G(1, t) = z_0$ for all t.

(Those of you who've studied this before will note that I'm not requiring D to be connected in order to be simply connected. Other definitions of simple connectivity may assume that D is connected.)

That's the formal definition. The important thing to remember about the formal definition is that in this course we pretty much *never* use it. Instead, it's important to remember two facts about simple connectivity. First, *star-shaped regions are simply connected.* Those are the simplest and easiest to construct examples of simply connected regions in the plane. Second, the following statements are all equivalent:

1. D is simply connected.
2. For each z_0 in $\mathbb{C} \setminus D$, the function $\log(z - z_0)$ can be well-defined on D, i.e. an analytic branch of the function exists.
3. For all loops γ contained in D and for all $z_0 \in \mathbb{C} \setminus D$, $W(\gamma, z_0) = 0$.
4. $\mathbb{C}^* \setminus D$ is a connected subset of $\mathbb{C}^*$, where $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere.

Number 4 is generally the easiest way to prove a region is simply connected, if it isn't star-shaped. Number 3 is usually the easiest way to show something *isn't* simply connected, e.g. an annulus or a punctured disk. And numbers 2 and 3 are generally the properties of simply connected regions that we're interested in for this course (but see also the material on analytic continuation, below).

Finally, there's one item that I've omitted from the above list that appears in Gamelin: "Every closed differential on D is exact". We did not spend much time talking about closed and exact differentials in this course, but if you're familiar with the terms already then it's an important thing to keep in mind.

6 Week 6

6.1 Review of power series

Recall the basic facts and definitions of power series: a series $\sum_n a_n$ of complex numbers *converges* if the sequence of partial sums converge. The series $\sum_n a_n$ *converges absolutely* if the series $\sum_n |a_n|$ of absolute values converges; absolute convergence implies convergence. A series which converges but does not converge absolutely is said to converge conditionally.

A series of functions $\sum_n f_n(z)$ defined on a domain D is said to *converge pointwise* if $\sum_n f_n(z_0)$ converges for each $z_0 \in D$. But pointwise convergence is not very useful; a pointwise limit of continuous functions isn't even necessarily continuous. A stronger form of convergence is *uniform convergence*:

a series of functions $\sum_n f_n(z)$ is said to *converge uniformly* to $f(z)$ on a domain D if for every $\epsilon > 0$ there exists an integer N such that for all $z \in D$, $|f(z) - \sum_{n=1}^{N} f_n(z)| < \epsilon$. Roughly speaking, $\sum_n f_n(z)$ converges uniformly on D if it converges at the same "speed" at every point in D. Uniform convergence implies pointwise convergence but has many more nice properties; for example the uniform limit of continuous functions is continuous, and the integral of a uniform limit of functions over a bounded set is equal to the limit of the integrals.

A *power series* centred at z_0 is just a series of functions of the form $\sum_n a_n(z - z_0)^n$, where the coefficients a_n are constants. A power series has a unique *radius of convergence* $R \in [0, \infty]$ with the following properties:

- The series converges absolutely if $|z - z_0| < R$.
- The series diverges if $|z - z_0| > R$.
- If $0 \leq \rho < R$, then the series converges uniformly on the disk $\{|z - z_0| \leq \rho\}$.

Note that R may equal ∞, in which case the series converges absolutely everywhere and converges uniformly on every compact set, or R may equal 0, in which case the series diverges everywhere except the point $z = z_0$ itself. If R is finite and not 0, then the behaviour on the circle $\{|z - z_0| = R\}$ is undetermined.

The radius of convergence can sometimes be determined by two tests: the Ratio Test and the Root Test. The *Ratio Test* says that if $|a_k/a_{k+1}|$ approaches a limit as $k \to \infty$, then $R = \lim |a_k/a_{k+1}|$. This test works even if the limit is ∞. The *Root Test* says that if $\sqrt[k]{|a_k|}$ approaches a limit as $k \to \infty$, then R is the inverse of that limit, i.e.

$$R = \frac{1}{\lim_{k\to\infty} \sqrt[k]{|a_k|}}.$$

This test works even if the limit is ∞ or 0 (in which case R is 0 or ∞ respectively). The *Cauchy-Hadamard* formula is a stronger version of the Root Test, in which "lim" is replaced by "lim sup", and is occasionally useful.

6.2 Cauchy's Integral Formula and Power Series

Cauchy's Integral Formula has several useful consequences when applied to power series and to series of functions in general. First, consider a series of analytic functions $\sum_n f_n(z)$ which converges uniformly to $f(z)$ on a domain D. Then in particular $\sum_n f_n(z)$ converges uniformly on any circle contained in D. Since the integral of a uniform limit of functions over a bounded set is equal to the the limit of the integrals, for any $z_0 \in D$ we get:

$$\begin{aligned}
\frac{1}{2\pi i}\int_{|z-z_0|<\epsilon} \frac{f(z)}{z-z_0}\,dz &= \sum_n \frac{1}{2\pi i}\int_{|z-z_0|<\epsilon} \frac{f_n(z)}{z-z_0}\,dz \\
&= \sum_n f_n(z_0) \\
&= f(z_0).
\end{aligned}$$

In other words, the Cauchy Integral Formula also holds for the limit. By differentiating under the integral sign we can show that $f(z)$ is therefore analytic; in other words *the uniform limit of analytic functions is analytic.* Note that in the real case the uniform limit of differentiable functions is not necessarily differentiable; this is another property that distinguishes analytic functions from real differentiable functions. By a similar argument we can show that *the derivative of the uniform limit of analytic functions is the uniform limit of the derivatives.* We already know that the same thing is true for the integrals over bounded sets.

We can also apply the Cauchy Integral Formula to power series. Suppose $f(z)$ is an analytic function on the domain $\{|z-z_0| < \rho\}$ for some $\rho > 0$. Start with Cauchy's Integral Formula, for any z satisfying $|z-z_0| < r$ where $r < \rho$:

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta-z_0|=r} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Re-write the integrand on the left-hand side using a power series centred at z_0:

$$\frac{f(\zeta)}{\zeta - z} = f(\zeta)\sum_{k=0}^{\infty} \frac{(z-z_0)^k}{(\zeta-z_0)^{k+1}}.$$

Note that since $|z-z_0| < r$, this power series converges uniformly with respect to ζ on the set $|\zeta - z_0| = r$ by the Ratio Test. Hence we can substitute it back into Cauchy's Integral Formula and swap the order of the sum and the integral to get

$$f(z) = \sum a_k (z-z_0)^k,$$

where

$$a_k = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta = \frac{f^{(k)}(z_0)}{k!}.$$

In other words, *the Taylor series for $f(z)$ converges to $f(z)$ for all z such that $|z - z_0| < r$*. And since this is true for all $r < \rho$, the Taylor series converges for all z such that $|z - z_0| < \rho$. This in turn implies that the radius of convergence of the Taylor series must be at least ρ, and that in turn implies that the Taylor series converges *uniformly* to $f(z)$ on the disk $\{|z - z_0| \leq r\}$ for all $r < \rho$.

In short, a function which is analytic on a disk is in fact the limit of its Taylor series on that disk, and the uniform limit of that series on any smaller disk. This is again not the case with real functions; the function $f(x) = e^{-1/x^2}$ if $x \neq 0$, $f(0) = 0$ is a smooth function defined on all of $\mathbb{R}$ whose Taylor series centred at 0 is the series $\sum_n 0$, which clearly doesn't converge to f anywhere except at $x = 0$.

Going the other way, a power series with centre z_0 and radius of convergence R converges to an analytic function defined on the disk $\{|z - z_0| < R\}$, and that convergence is uniform on any smaller disk.

More generally, if $f(z)$ is an analytic function on a domain D which is not a disk and $z_0 \in D$, the Taylor series centred at z_0 will have a radius of convergence equal to the largest disk centred at z_0 contained in D. Typically this equals the distance from z_0 to the nearest non-removable singularity of $f(z)$. For example, the Taylor series of $1/(z^2+1)$ centred at 0 is $1 - z^2 + z^4 - z^6 + \cdots$, which has radius of convergence 1 by the Cauchy-Hadamard test, and 1 is the distance from 0 to the nearest singularities of the function, at $\pm i$. Similarly, the Taylor series of the same function centred at 1, whatever it is, must have radius of convergence equal to $\sqrt{2}$.

6.3 Zeroes of analytic functions

An important consequence of the connection between analytic functions and power series is the fact that if $f(z)$ is a non-constant analytic function on a connected domain D, the zeroes of $f(z)$ must be *isolated*; in other words, for every $z_0 \in D$ such that $f(z_0) = 0$ there exists a radius $\epsilon > 0$ such that no other zeroes of f lie in the disk $\{|z - z_0| < \epsilon\}$.

First, recall that z_0 is a *zero of order N* if $f^{(N)}(z_0) \neq 0$, but $f^{(k)}(z_0) = 0$ for all $k = 0, 1, \ldots, N-1$. If z_0 is a zero of order N then f has a power series expansion of the form

$$f(z) = \sum_{k=N}^{\infty} a_k (z - z_0)^k,$$

where $a_N \neq 0$. Thus there exists an analytic function $h(z)$ defined on $\{|z - z_0| < \epsilon\}$ for some $\epsilon > 0$ such that $f(z) = (z - z_0)^N h(z)$, and $h(z_0) \neq 0$. By continuity $h(z) \neq 0$ for all z is some smaller neighbourhood of z_0, and hence z_0 is the only zero of f in that neighbourhood. Thus z_0 is an isolated zero of f.

The other possibility is that $f(z_0) = 0$ and $f^{(k)}(z_0) = 0$ for all k. Then $\sum_n 0$ is a power series expansion for f near z_0, and this expansion must converge to f in some disk centred at z_0; in other words there exists an open neighbourhood of z_0 in D such that $f(z) = 0$ everywhere in that neighbourhood. Let A be the set of all points z such that $f^{(k)}(z) = 0$ for all k; by the above argument A is an open subset of D. Let $B = D \setminus A$; B is the set of points where $f^{(k)}(z) \neq 0$ for some k. This is also an open subset of D; it's the union of the sets $B_k = \{z | f^{(k)}(z) \neq 0\}$, each of which is open by continuity. Since D is connected, one of A or B must be empty and the other must be all of D; since $z_0 \in A$ by assumption, we must have $A = D$, that is $f(z) = 0$ for all $z \in D$. Since f was assumed to be non-constant, this is a contradiction, completing the proof. Note that while this proof requires D to be connected, the result holds true when D is not connected, provided that we assume f is not constant on any component of D.

A consequence of this is that if $f(z)$ is an analytic function and the set $\{z | f(z) = 0\}$ has even a single limit point, then f must be the zero function. Similarly if $f(z)$ and $g(z)$ are two analytic functions and the set $\{z | f(z) = g(z)\}$ has a limit point, then $f = g$.

6.4 Analytic Continuation

Finally, we can use power series to extend the domain of analytic functions in certain cases. Suppose f is an analytic function on a domain D and $\gamma : [a, b] \to \mathbb{C}$ is a path starting at $z_0 \in D$ whose other endpoint may or may not be in D. Then we say that f is *analytically continuable along* γ if there exist functions $r(t)$ and $a_n(t)$, $n = 0, 1, \ldots,$ defined on $[a, b]$ such that for each t,

$$f_t(z) = \sum_{n=0}^{\infty} a_n(t)(z - \gamma(t))^n$$

is a convergent power series on the set $\{|z - \gamma(t)| < r(t)\}$, with $f_a(z) = f(z)$ for all $|z - z_0| < r(a)$, and furthermore $f_s(z) = f_t(z)$ for all $z \in \{|z - \gamma(s)| < r(s)\} \cap \{|z - \gamma(t)| < r(t)\}$ whenever s is close to t.

In other words, we try to create a new function $f_t(z)$ for every point along the path γ, such that for nearby s and t the functions f_s and f_t actually equal one another on the intersection of their domains, so that f_t represents "the same function" at each step of the process. This condition implies that the functions $r(t)$ and $a_n(t)$ will be continuous functions of t, and in a sense what we are doing is letting the power series expansions of f tell us what f should equal outside of D. If we can construct such functions, the final function $f_b(z)$ is called the *analytic continuation of* $f(z)$ *along* γ. The analytic continuation of $f(z)$ along γ is not guaranteed to exist, but if it does exist then it is unique.

One very important question is how much does the analytic continuation depend on the choice of path? If γ_0 and γ_1 are two paths from z_0 to z_1, will the analytic continuations of $f(z)$ along both paths be equal? Not always, but under certain circumstances we have the *Monodromy Theorem*. If we can *deform* γ_0 to γ_1, that is if there exists a continuous function $G(s,t)$ from $[0,1] \times [a,b]$ to $\mathbb{C}$ such that $G(0,t) = \gamma_0(t)$, $G(1,t) = \gamma_1(t)$, and such that for each $s \in [0,1]$ the path $\gamma_s(t) = G(s,t)$ is a path from z_0 to z_1 and f can be analytically continued along each γ_s... then yes, the continuations of f along γ_0 and γ_1 will agree at z_1.

But the cases where the analytic continuations don't agree are in many ways more interesting. For example, the principal branch of $f(z) = \sqrt{z}$ defined on the right half-plane has a series expansion $f(z) = 1 + \frac{1}{2}(z-1) - \frac{1}{8}(z-1)^2 + \cdots$ at $z = 1$. If we let

$$f_t(z) = e^{it/2} + \frac{e^{-it/2}}{2}(z - e^{it}) - \frac{e^{-3it/2}}{8}(z - e^{it})^2 + \cdots,$$

we obtain an analytic continuation of $f(z)$ around the unit circle $z = e^{it}$. If we follow the unit circle counterclockwise from $t = 0$ to $t = \pi$, we get the following continuation of $f(z)$ at -1:

$$f_\pi(z) = i + \frac{-i}{2}(z+1) - \frac{i}{8}(z+1)^2 + \cdots.$$

But if we follow the unit circle clockwise from $t = 0$ to $t = -\pi$, we get the following continuation of $f(z)$ at -1:

$$f_{-\pi}(z) = -i + \frac{i}{2}(z+1) - \frac{-i}{8}(z+1)^2 + \cdots ,$$

which is the other branch of the square root function.