Chapter 3: Multiple Random Variables

CLO3	Define multiple random variables in terms of their PDF and CDF and calculate joint moments such as the correlation and covariance.

Objectives

1. **Introduce Joint distribution, joint density, conditional distribution and density, Statistical independence**
2. **Introduce Expectations, correlations and joint characteristic functions**

1. Vector Random Variables

- Let two random variables X with value x and Y with value y are defined on a sample space S, then the random point (x, y) is a random vector in the XY plane.
- In the general case where N r.v's. $X_1, X_2, \dots X_N$ are defined on a sample space S, they become N-dimensional random vector or N-dimensional r.v.

2. Joint distribution

- Let two events $A = \{X \leq x\}\ and\ B = \{Y \leq y\}$, (*X* and *Y* two random variables) with probability distribution functions $F_X(x)$ and $F_Y(y)$, respectively:

$$F_X(x) = P(X \leq x) \quad (1)$$

$$F_Y(y) = P(Y \leq y) \quad (2)$$

- The probability of the joint event $\{X \leq x, Y \leq y\}$, which is a function of the members x and y is called the joint probability distribution function $P\{X \leq x, Y \leq y\} = P(A \cap B)$. It is given as follows:

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y) \quad (3)$$

- If *X* and *Y* are two discrete random variables, where *X* have *N* possible values x_n and *Y* have *M* possible values y_m, then:

$$F_{X,Y}(x, y) = \sum_{n=1}^{N} \sum_{m=1}^{M} P(x_n, y_m) u(x - x_n) u(y - y_m) \quad (4)$$

Probability of the joint event $\{X = x_n, Y = y_m\}$

Unit step function

- If *X* and *Y* are two continues random variables, then:

$$F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(u,v)dudv \quad (5)$$

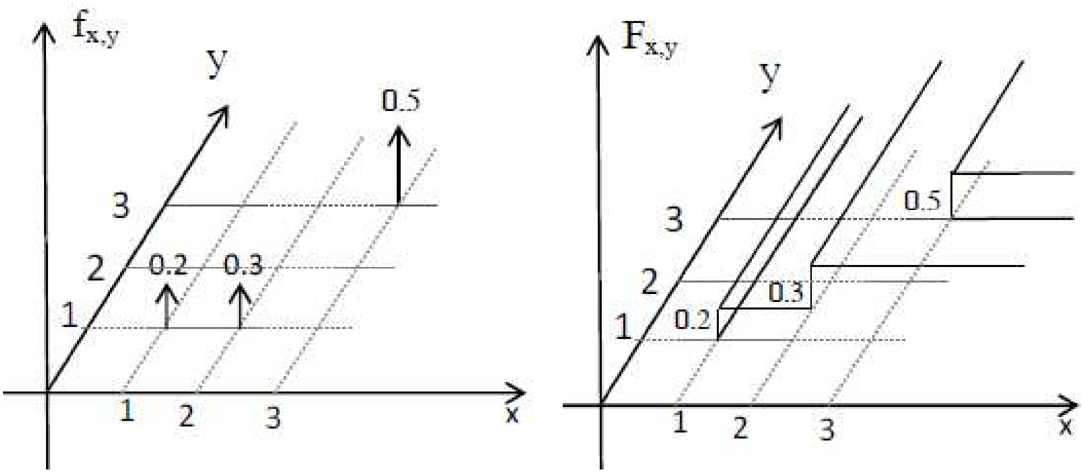
Example 1:

Let *X* and *Y* be two discrete random variables. Let us assume that the joint space has only three possible elements (1, 1), (2, 1) and (3, 3). The probabilities of these events are:

$P(1,1) = 0.2,\ P(2,1) = 0.3,\ and\ P(3,3) = 0.5.$

Find and plot $F_{X,Y}(x,y)$.

Solution:



Proprieties of the joint distribution $F_{X,Y}(x,y)$:

1. $F_{X,Y}(-\infty,-\infty) = F_{X,Y}(-\infty,y) = F_{X,Y}(x,-\infty) = 0$
2. $F_{X,Y}(\infty,\infty) = 1$
3. $0 \le F_{X,Y}(x,y) \le 1$
4. $P\{x_1 < X \le x_2,\ y_1 < Y \le y_2\} = F_{X,Y}(x_2,y_2) - F_{X,Y}(x_1,y_2) - F_{X,Y}(x_2,y_1) + F_{X,Y}(x_1,y_1)$
5. $F_{X,Y}(x,y)$ is a non decreasing function of both x and y

Marginal distributions:

- The marginal distribution functions of one random variable is expressed as:
  - ✓ $\boldsymbol{F_{X,Y}(x,\infty) = F_X(x)}$
  - ✓ $\boldsymbol{F_{X,Y}(\infty,y) = F_Y(y)}$

Example 2:

$\boldsymbol{S} = \{(1,1),(2,1),(3,3)\}$

$P(1,1) = 0.2, \qquad P(2,1) = 0.3, \qquad P(3,3) = 0.5$

Find $F_{X,Y}(x,y)$ and the marginal distributions $F_X(x)$ and $F_Y(y)$ of example 1

Solution:

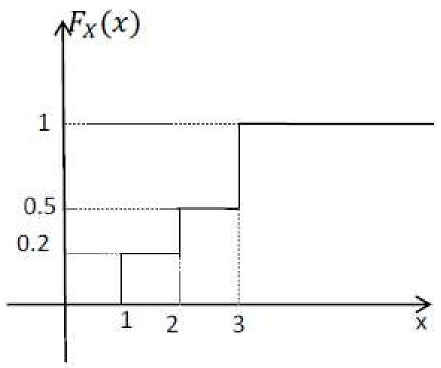
$F_{X,Y}(x,y) = P(1,1)\,u(x-1)\,u(y-1) + P(2,1)\,u(x-2)\,u(y-1) + P(3,3)\,u(x-3)\,u(y-3)$

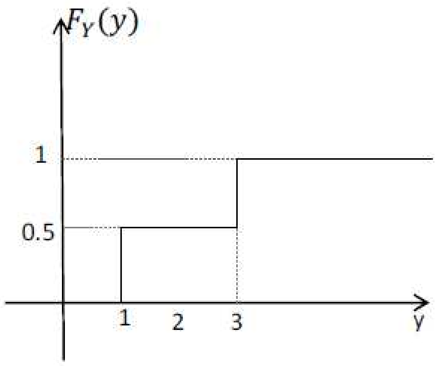
$= 0.2\,u(x-1)\,u(y-1) + 0.3\,u(x-2)\,u(y-1) + 0.5\,u(x-3)\,u(y-3)$

$$F_X(x) = F_{X,Y}(x,\infty) = 0.2\,u(x-1) + 0.3\,u(x-2) + 0.5\,u(x-3)$$

$$F_Y(y) = F_{X,Y}(\infty,y) = 0.2\,u(y-1) + 0.3\,u(y-1) + 0.5\,u(y-3)$$

$$= 0.5\,u(y-1) + 0.5\,u(y-3)$$





3. Joint density

- For two continuous random variables *X* and *Y,* the joint probability density function is given by:

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} \quad (6)$$

- If *X* and *Y,* are tow discrete random variables then the joint probability density function is given by:

$$f_{X,Y}(x,y) = \sum_{n=1}^{N}\sum_{m=1}^{M} P(x_n, y_m)\delta(x - x_n)\delta(y - y_m) \quad (7)$$

Proprieties of the joint density function $f_{X,Y}(x,y)$:

1. $f_{X,Y}(x,y) \geq 0$
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$
3. $F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(x,y)dxdy$
4. $F_X(x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dydx,\ F_Y(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy$
5. $P\{x_1 < X \leq x_2\ , y_1 < Y \leq y_2\} = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{X,Y}(x,y)dxdy$
6. $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy\ ,\ f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$ (Marginal density function)

Example 3:

Let us consider the following joint probability density function of two random variables *X* and *Y*:

$$f_{X,Y}(x,y) = u(x)u(y)xe^{-x(y+1)}$$

Find the marginal probability density functions $f_X(x)$ and $f_Y(y)$

Solution:

$$f_X(x) = \int_0^{\infty} u(x)\, x\, e^{-x(y+1)} dy = u(x) x\, e^{-x} \frac{e^{-xy}}{-x} |_0^{\infty} = u(x) e^{-x}$$

$$f_Y(y) = \int_0^{\infty} u(y)\, x\, e^{-x(y+1)} dx$$

But $\int x e^{ax} dx = e^{ax} [\frac{x}{a} - \frac{1}{a^2}]$

$$f_Y(y) = u(y)[e^{-x(y+1)} (\frac{-x}{(y+1)} - \frac{1}{(y+1)^2}]_0^{\infty} = \frac{u(y)}{(y+1)^2}$$

4. Conditional distribution and density

- The conditional distribution function of a random variable *X*, given event *B* with $P(B) \neq 0$ is:

$$F_X(x|B) = P\{X \leq x|B\} = \frac{P\{X \leq x \cap B\}}{P(B)} \quad (8)$$

- The corresponding conditional density function is:

$$f_X(x|B) = \frac{dF_X(x|B)}{dx} \quad (9)$$

- Often, we are interested in computing the distribution function of one random variable *X* conditioned by the fact that the second variable has some specific values.

4.1 Point conditioning: The Radom variable has some specific value

- For continuous random variables:

$$f_X(x|\mathrm{y}) = \frac{f_{X,Y}(x,y)}{f_{Y,}(y)} \quad (10)$$

Also, we have:

$$f_Y(y|x) = \frac{f_{X,Y}(x,y)}{f_{X,}(x)} \tag{11}$$

- For discrete random variables:

Suppose we have:

$$f_Y(\text{y}) = \sum_{j=1}^{M} P(y_j)\delta(y - y_j) \tag{12}$$

$$f_{X,Y}(x,y) = \sum_{i=1}^{N}\sum_{j=1}^{M} P\left(x_i, y_j\right)\delta(x - x_i)\,\delta\left(y - y_j\right) \tag{13}$$

Assume $y = y_k$ is the specific value of y, then:

$$F_X(x|\,Y = y_k) = \sum_{i=1}^{N} \frac{P(x_i, y_k)}{P(y_k)}\, u(x - x_i) \tag{14}$$

and

$$f_X(x|\,Y = y_k) = \sum_{i=1}^{N} \frac{P(x_i, y_k)}{P(y_k)}\, \delta(x - x_i) \tag{15}$$

Example 4:

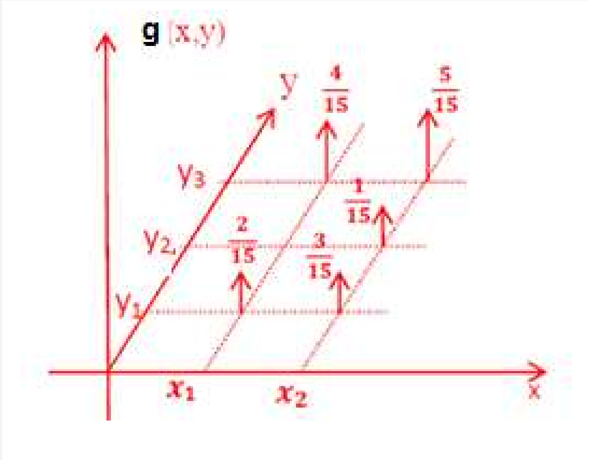
Let us consider the joint pdf: $f_{X,Y}(x,y) = u(x)u(y)x\, e^{-x(y+1)}$

Find $f_Y(y|x)$ if the marginal pdf of *X* is given by: $f_X(x) = u(x)e^{-x}$

Solution:

$f_{X,Y}(x,y)$ and $f_X(x)$ are nonzero only for $y > 0$ *and* $x > 0$, $f_Y(y|x)$ is nonzero only for $y > 0$ *and* $x > 0$, therefore, we keep u(x).

$$f_Y(y|x) = \frac{f_{X,Y}(x,y)}{f_{X,}(x)} = u(x)u(y)x\, e^{-xy}$$



Example 5:

What represents the corresponding figure ?

Find $f_x(x|y=y_3)$

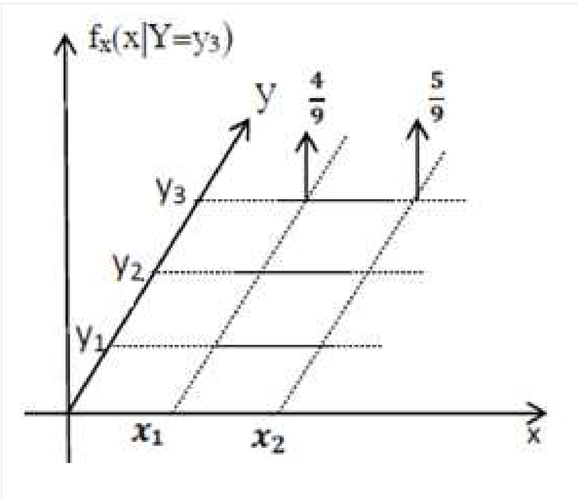
Solution:

$$f_x(x|Y=y_3)=\sum_{i=1}^{2}\frac{P(x_i,y_3)}{P(y_3)}\delta(x-x_i)$$

$$P(y_3)=\frac{4}{15}+\frac{5}{15}=\frac{9}{15}$$

$$f_X(x|Y=y_3)=\frac{P(x_1,y_3)}{9/15}\delta(x-x_1)+\frac{P(x_2,y_3)}{9/15}\delta(x-x_2)$$

$$=\frac{4}{9}\delta(x-x_1)+\frac{5}{9}\delta(x-x_2)$$



4.2 Interval conditioning: The Radom variable in the interval $\{y_a < Y \leq y_b\}$

$y_a and\ y_b$ are real numbers.

$$f_x(x|y_a < Y \leq y_b) = \frac{\int_{y_a}^{y_b} f_{X,Y}(x,y)dy}{\int_{y_a}^{y_b} f_Y(y)dy} = \frac{F_{X,Y}(x,y_b) - F_{X,Y}(x,y_a)}{\int_{y_a}^{y_b} f_Y(y)dy} \quad (16)$$

and

$$F_x(x|y_a < Y \leq y_b) = \frac{\int_{y_a}^{y_b}\int_{-\infty}^{x} f_{X,Y}(u,v)dudv}{\int_{y_a}^{y_b}\int_{-\infty}^{\infty} f_{X,Y}(u,v)dudv} \quad (17)$$

Example 6:

Let $f_{X,Y}(x,y) = u(x)u(y)x\ e^{-x(y+1)}$ (18)

$f_X(x) = u(x)e^{-x}$, *and* $f_Y(y) = \frac{u(y)}{(y+1)^2}$

Find $f_X(x|Y \leq y)$?

Solution 6:

$$f_x(x|y \leq y) = f_x(x|-\infty < y \leq y) = \frac{\int_{-\infty}^{y} f_{X,Y}(x,y)dy}{\int_{-\infty}^{y} f_Y(y)dy}$$

$$\int_{-\infty}^{y} f_{X,Y}(x,y)dy = \int_0^y u(x)xe^{-x(y+1)}dy$$

$$= u(x)x\,e^{-x}.\frac{e^{-xy}}{-x}|_0^y = -u(x)e^{-x}[e^{-xy}-1]$$

$$= u(x)e^{-x}[1-e^{-xy}] \qquad y>0$$

$$\int_{-\infty}^{y} f_Y(y)dy = \int_0^y \frac{1}{(y+1)^2}dy = \frac{-1}{y+1}|_0^y = \frac{-1}{y+1}+1 = \frac{y}{y+1} \qquad y>0$$

Then, $f_X(x|Y \le y) = u(x)\,u(y)e^{-x}(1-e^{-xy})\,(\frac{y+1}{y})$

5. Statistical independence

- Two random variables X and Y are independent if:

$$F_{X,Y}(x,y) = F_X(x)F_Y(y) \tag{19}$$

  This means that

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) \tag{20}$$

- Note that if X and Y are independent, then

$$f_X(x|y) = f_X(x) \;\; and \;\; f_X(y|x) = f_y(y) \tag{21}$$

Example 7:

In example 6, are *X* and *Y* independent?

Solution

$$f_{X,Y}(x,y) = u(x)u(y)x\,e^{-x(y+1)}$$

$$f_X(x)f_Y(y) = u(x)u(y)e^{-x}.\frac{1}{(y+1)^2} \neq f_{X,Y}(x,y)$$

$\Longrightarrow X \ and \ Y \ are \ not \ independent.$

Example 8:

Let $f_{X,Y}(x,y) = \frac{1}{12}u(x)u(y)e^{-\frac{x}{4}-\frac{y}{3}}$ are X and Y independent?

Solution

$$f_X(x) = \int_0^{\infty} \frac{1}{12}\, u(x)e^{-\frac{x}{4}-\frac{y}{3}}dy \ = \frac{u(x)}{12}e^{-\frac{x}{4}} . -3e^{-\frac{y}{3}}\Big|_0^{\infty} = \frac{u(x)}{4}e^{\frac{-x}{4}}$$

$$f_Y(y) = \int_0^{\infty} \frac{1}{12}\, u(y)e^{-\frac{x}{4}-\frac{y}{3}}dx = \frac{u(y)}{3}e^{\frac{-y}{3}}$$

$$f_X(x)f_Y(y) = \frac{1}{12}\, u(x)u(y)e^{-\frac{x}{4}-\frac{y}{3}} = f_{X,Y}(x,y) \Rightarrow X \ \&Y \text{ are independent.}$$

6. Sum of two random variables

- Here the problem is to determine the probability density function of the sum of two ***independent*** random variables *X* and *Y*:

$$W = X + Y \tag{22}$$

- The resulting probability density function of W can be shown to be the convolution of the density functions of *X* and *Y:*

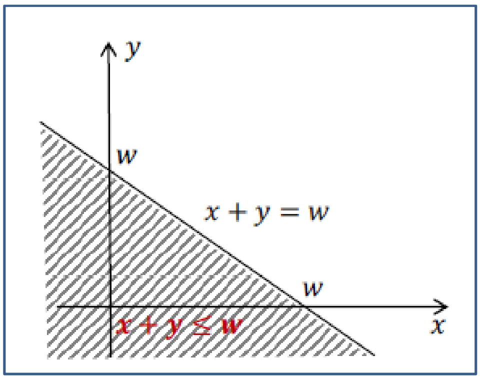
Convolution

$$f_W(w) = f_x(x) * f_Y(y)$$

$$f_W(w) = \int_{-\infty}^{\infty} f_Y(y)f_X(w-y)dy = \int_{-\infty}^{\infty} f_X(x)f_Y(w-x)dx \tag{23}$$

Proof:

- Let W equals the sum of two independent r.v.s X and Y



$$W = X + Y$$

Then, $F_W(w) = P\{W \leq w\} = P\{X + Y \leq w\}$

$X + Y \leq w$ corresponds to the shaded area, therefore:

$$F_W(w) = \int_{-\infty}^{\infty} \int_{-\infty}^{x=w-y} f_{X,Y}(x,y)dxdy$$

Since X and Y are independent:

$$F_W(w) = \int_{-\infty}^{\infty} \int_{-\infty}^{w-y} f_Y(y)\, f_X(x)dxdy$$

$$= \int_{-\infty}^{\infty} f_Y(y) \int_{-\infty}^{w-y} f_X(x)dxdy$$

$$= \int_{-\infty}^{\infty} f_Y(y)F_X(w-y)dy$$

Differentiating w.r.t. w:

$$f_W(w) = \int_{-\infty}^{\infty} f_Y(y)f_X(w-y)dy = f_Y(w) * f_X(w)$$

So, the density function of the sum of two independent r.v.s is the convolution of their density functions.

Example 9:

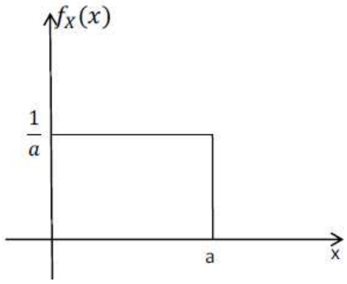
Let us consider two independent random variables X and Y with the following pdfs:

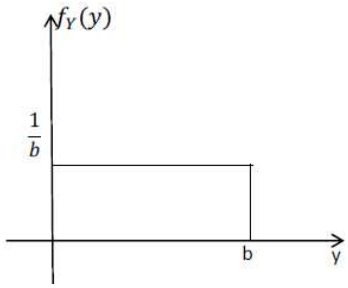
$f_X(x) = \frac{1}{a}[u(x) - u(x-a)]$

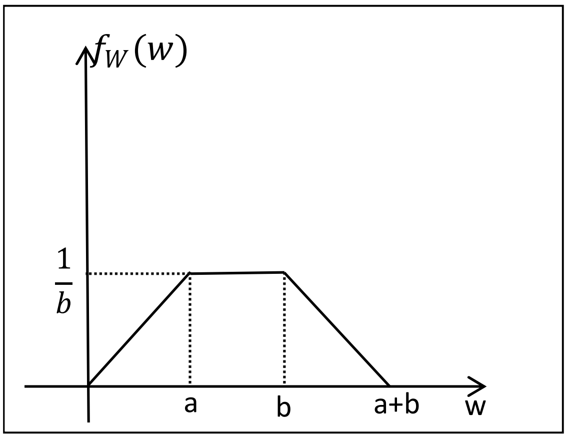
$f_Y(y) = \frac{1}{b}[u(y) - u(y-b)]$ Where $0 < a < b$

Find the pdf of $W=X+Y$

Solution:







7. Central Limit Theorem

The probability distribution function of the sum of a large number of random variables approaches a Gaussian distribution.

8. Expectations and Correlations

- If $\boldsymbol{g(x,y)}$ is a function of two random variables X and Y (their joint probability density function $\boldsymbol{f_{X,Y}(x,y)}$). Then the expected value of $\boldsymbol{g(x,y)}$, a function of the two random variables X and Y is given by:

$$\boldsymbol{E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dxdy} \quad (24)$$

Example 10

Let $g(X,Y) = aX + bY$, find $E[g(X,Y)]$?

Solution:

$$\bar{g} = E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(aX + bY) f_{X,Y}(x,y)dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ax\, f_{X,Y}(x,y)dxdy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} by\, f_{X,Y}(x,y)dxdy$$

$$= a\int_{-\infty}^{\infty} x\, f_X(x)dx + b\int_{-\infty}^{\infty} y\, f_Y(y)dy$$

$$= aE[X] + bE[Y]$$

- If we have n functions of random variables $\boldsymbol{g_1(x,y), g_2(x,y), \ldots, g_n(x,y)}$ **then**:

$$\boldsymbol{E[g_1(x,y) + g_2(x,y) + \ldots + g_n(x,y)] = E[g_1(x,y)] + E[g_2(x,y)] + \cdots + E[g_n(x,y)]}$$

  Which means that the expected value of the sum of the functions is equal to the sum of the expected values of the functions.

8.1 Joint Moments about the origin

$$m_{nk} = E[X^n Y^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k\, f_{X,Y}(x,y)dxdy \tag{25}$$

We have the following proprieties:

- ✓ $m_{n0} = E[X^n]$ are the moments of X
- ✓ $m_{0k} = E[Y^k]$ are the moments of Y
- ✓ The sum ***n+k*** is called the **order of the moments**. (m_{20}, m_{02} and m_{11} are called second order moments).
- ✓ $m_{10} = E[X]$ and $m_{01} = E[Y]$ are called first order moments.

8.2 Correlation

- The second-order joint moment m_{11} is called the correlation between *X* and *Y*.
- The correlation is a very important statistic and its denoted by R_{XY}:

$$R_{XY} = E[XY] = m_{11} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, y\, f_{X,Y}(x,y)dxdy \tag{26}$$

- If $R_{XY} = E[X]E[Y]$ then *X* and *Y* are said to be uncorrelated.
- If *X* and *Y* are independent then $f_{X,Y}(\mathrm{x},\mathrm{y}) = f_X(x)f_Y(y)$ and

$$R_{XY} = \int_{-\infty}^{\infty} x f_X(x)d_x \int_{-\infty}^{\infty} y f_Y(y)dy = E[X]E[Y] \tag{27}$$

- Therefore, if *X* and *Y* are independent *then* they are uncorrelated.
- However, if X and Y are uncorrelated, it is not necessary that they are independent.

- *If* $R_{XY} = 0$ *then* X *and* Y are called orthogonal.

Example 11

Let $E[X] = 3, {\sigma_x}^2 = 2,$ let also $Y = -6X + 22$

Find R_{XY}?

Are X and Y Orthogonal?

Are X and Y uncorrelated?

Solution:

R_{XY}=E[XY]=E[X(-6X+22)]

$= E[-6X^2] + E[22X]$ we know ${\sigma_x}^2 = E[X^2] - \bar{X}^2$

$E[X^2] = {\sigma_x}^2 + \bar{X}^2 = 11$

$R_{XY} = -6(11) + 22(3) = 0$ ⇒ X and Y are Orthogonal

$E[Y] = -6E[X] + 22 = -6(3) + 22 = 4$ ⇒ E[X] E[Y] = 12

$R_{XY} \neq E[X]E[Y]$ ⇒X and Y are not uncorrelated.

8.3 Joint central moments

- The joint central moment is defined as:

$$u_{nk} = E[(X - \bar{X})^n (Y - \bar{Y})^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - \bar{X})^n (y - \bar{Y})^k f_{X,Y}(x, y)dxdy \quad (28)$$

We note that:

$u_{20} = E[(X - \bar{X})^2] = {\sigma_X}^2$ is the variance of X.

$u_{o2} = E[(Y - \bar{Y})^2] = \sigma_Y{}^2$ is the variance of Y.

8.4 Covariance

- The second order joint central moment $\boldsymbol{u_{11}}$ is called the covariance of X and Y and denoted by C_{XY}.

$$C_{XY} = u_{11} = E[(X - \bar{X})(Y - \bar{Y})] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x - \bar{X})(y - \bar{Y})\, f_{X,Y}(x, y)dxdy \qquad (29)$$

We have

$$C_{XY} = E[XY] - \bar{X}\bar{Y} = R_{XY} - E[X]E[Y] \qquad (30)$$

- If X and Y are uncorrelated (or independent), then $C_{XY} = 0$
- If X and Y are orthogonal, then $C_{XY=} - \bar{X}\bar{Y}$
- If X and Y are correlated, then the ***correlation coefficient*** $\boldsymbol{\rho}$ measures the degree of correlation between X and Y:

$$\rho = \frac{\mu_{11}}{\sqrt{\mu_{20}\mu_{02}}} = \frac{C_{XY}}{\sigma_X \sigma_Y} = E[\left(\frac{X-\bar{X}}{\sigma_X}\right)\left(\frac{Y-\bar{Y}}{\sigma_Y}\right)] \qquad (31)$$

It is important to note that: $-1 \leq \rho \leq 1$

Example 12

Let $g = aX + bY$ find $\sigma_g{}^2$ when X and Y are uncorrelated

Solution:

$\sigma_g{}^2 = E[g^2] - E[g]^2$

$E[g^2] = E[(aX + bY)^2] = a^2E[X^2] + 2abE[XY] + b^2E[Y^2]$

$E[g]^2 = (aE[X] + b[Y])^2 = a^2\bar{X}^2 + 2ab\bar{X}\bar{Y} + b^2\bar{Y}^2$

$\sigma_g{}^2 = a^2(E[X^2] - \bar{X}^2) + 2ab(E[XY] - \bar{X}\bar{Y}) + b^2(E[Y^2] - \bar{Y}^2)$

$= a^2\sigma_X{}^2 + 2abC_{XY} + b^2\sigma^2{}_Y$

$if\ X\ and\ Y\ are\ uncorrelated,\ C_{XY} = 0 \ => \sigma_g{}^2 = a^2\sigma_X{}^2 + b^2\sigma_Y{}^2$

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9 Joint Characteristic functions

- The joint characteristic function of two random variables is defined by:

$$\emptyset_{X,Y}(\omega_1, \omega_2) = E\left[e^{j\omega_1 X + j\omega_2 Y}\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{e^{j\omega_1 x + j\omega_2 y}} f_{X,Y}(x, y)dxdy \quad (32)$$

ω_1 and ω_2 are real numbers.

- By setting $\omega_1 = 0\ or\ \omega_2 = 0$, we obtain the marginal characteristic function:

$$\emptyset_X(\omega_1) = \emptyset_{X,Y}(\omega_1, 0)$$
$$\emptyset_Y(\omega_2) = \emptyset_{X,Y}(0, \omega_2) \quad (33)$$

- The joint moments are obtained as :

$$m_{nk} = (-j)^{n+k} \frac{\partial^{n+k}\emptyset_{X,Y}(\omega_1,\omega_2)}{\partial_{\omega_1}{}^{n}\ \partial_{\omega_1}{}^{k}} \mid_{\omega_1=0,\omega_2=0} \tag{34}$$

Example 1

Let us consider the joint characteristic function: $\emptyset_{X,Y}(\omega_1, \omega_2) = e^{-2\omega_1{}^2-8w_2{}^2}$

Find *E*[*X*] and *E*[*Y*]

Are *X* and *Y* uncorrelated?

Solution:

$$E[X^1Y^0] = \bar{X} = m_{10} = -j\frac{d\emptyset_{X,Y}(w_1, w_2)}{dw_1} \mid_{w_1=w_2=0}$$

$$= -j\left(-4w_1 e^{-2w_1{}^2-8w_2{}^2}\right) \mid_{w_1=0=w_2} = 0$$

$$E[X^0Y^1] = \bar{Y} = m_{01} = -j\left(-16w_2 e^{-2w_1{}^2-8w_2{}^2}\right)\mid_{w_1=w_2=0} = 0$$

$$R_{XY} = E[XY] = m_{11} = (-j)^2\frac{d^2\emptyset_{X,Y}(w_1, w_2)}{dw_1 dw_2} \mid_{w_1=w_2=0} = 0$$

$$= -(-4w_1)(-16w_2)e^{-2w_1{}^2-8w_2{}^2}\mid_{w_1=w_2=0} = 0$$

$R_{XY} = 0 = E[X]E[Y] \Rightarrow$ X and Y are uncorrelated.