2. Annuities

1. Basic Annuities

1.1 Introduction

Annuity: A series of payments made at equal intervals of time.

Examples: House rents, mortgage payments, installment payments on automobiles, and interest payments on money invested.

Annuity-certain: An annuity such that payments are certain to be made for a fixed period of time.

Term: The fixed period of time for which payments are made

Contingent annuity: An annuity under which the payments are not certain to be made. A common type of contingent annuity is one in which payments are made only if a person is alive (**Life Annuity**).

Payment period: Interval between annuity payments.

1.2. Annuity-immediate

Annuity-immediate: An annuity under which payments of 1 are made at the end of each period for n periods.

$a_{\overline{n}|}$: The present value of the annuity at one period before the first payment is made.

$$\begin{aligned} a_{\overline{n}|} &= v + v^2 + \cdots + v^{n-1} + v^n \\ &= v\frac{1-v^n}{1-v} = v\frac{1-v^n}{iv} \\ &= \frac{1-v^n}{i} \end{aligned}$$

$s_{\overline{n}|}$: The accumulated value of the annuity at n periods just after the last payment is made.

$$\begin{aligned} s_{\overline{n}|} &= 1 + (1+i) + \cdots + (1+i)^{n-2} + (1+i)^{n-1} \\ &= \frac{(1+i)^n - 1}{i} \end{aligned}$$

A verbal interpretation to the formula for $a_{\overline{n}|}$

$$1 = i a_{\overline{n}|} + v^n.$$

Relationship between $s_{\overline{n}|}$ and $a_{\overline{n}|}$.

$$s_{\overline{n}|} = a_{\overline{n}|}(1+i)^n$$

and

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$$

This relationship can be derived as follows:

$$\begin{aligned}
\frac{1}{s_{\overline{n}|}} + i &= \frac{i}{(1+i)^n - 1} + i \\
&= \frac{i + i(1+i)^n - i}{(1+i)^n - 1} = \frac{i}{1 - v^n} \\
&= \frac{1}{a_{\overline{n}|}}
\end{aligned}$$

Example 1 Find the present value of an annuity which pays \$500 at end of each half-year for 20 years if the rate of interest is 9% convertible semiannually.

The answer is

$$500a_{\overline{40}|.045} = 500(18.4016) = \$9200.80.$$

Example 2 If a person invests \$1000 at 8% per annum convertible quarterly, how much can be withdraw at end of every quarter to use up the fund exactly at the end of 10 years.

Let R be the amount of each withdrawal. The equation of value at the date of investment is

$$Ra_{\overline{40}|0.02} = 1000.$$

Thus, we have

$$R = \frac{1000}{a_{\overline{40}|0.02}} = \frac{1000}{27.3555} = \$36.56.$$

Example 3 Compare the total amount of interest that would be paid on a \$1000 loan over 10-year period, if the effective rate of interest is 9% per annum, under the following three repayment methods: (1) The entire loan plus accumulated interest is paid in one lump-sum at end of 10 years. (2) Interest is paid each each year as accrued and the principal is repaid at the end of 10 years. (3) The loan is repaid by level payments over the 10-years.
(1) The accumulated value of the loan at the end of 10 years is

$$1000(1.09)^{10} = \$2367.36.$$

Thus the total amount of interest paid is equal to

$$\$2367.36 - 1000.00 = \$1367.36.$$

(2)Each year the loan earns interest of $1000(.09) = \$90$, so that the total amount of interest paid is equal to

$$10 \cdot 90 = \$900.00.$$

(3) Let the level payment be R. An equation of value for R at the inception of the loan is

$$Ra_{\overline{10|}} = 1000$$

which gives

$$R = \frac{1000}{a_{\overline{10|}}} = \frac{1000}{6.417658} = \$155.82.$$

Thus the total amount of interest paid is equal to

$$10(155.82) - 1000 = \$558.20.$$

1.3. Annuity-Due

Annuity-Due The payments are made at the beginning of the period.

The present value of the annuity $\ddot{a}_{\overline{n}|}$:

$$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + \cdots + v^{n-1} = \frac{1 - v^n}{1 - v} = \frac{1 - v^n}{iv} = \frac{1 - v^n}{d}$$

The accumulated value of the annuity $\ddot{s}_{\overline{n}|}$:

$$\begin{aligned} \ddot{s}_{\overline{n}|} &= 1 + i + (1+i)^2 + \cdots + (1+i)^{n-1} + (1+i)^n \\ &= (1+i)\frac{(1+i)^n - 1}{1 + i - 1} = \frac{(1+i)^n - 1}{iv} \\ &= \frac{(1+i)^n - 1}{d} \end{aligned}$$

Relationship between $\ddot{a}_{\overline{n}|}$,$\ddot{s}_{\overline{n}|}$

$$\ddot{s}_{\overline{n}|} = \ddot{a}_{\overline{n}|}(1+i)^n \quad \text{and} \quad \frac{1}{\ddot{a}_{\overline{n}|}} = \frac{1}{\ddot{s}_{\overline{n}|}} + d$$

Relationship between $\ddot{a}_{\overline{n}|}$, $\ddot{s}_{\overline{n}|}$ and $a_{\overline{n}|}$ and $s_{\overline{n}|}$

$$\ddot{a}_{\overline{n}|} = a_{\overline{n}|}(1+i) \quad \text{and} \quad \ddot{s}_{\overline{n}|} = s_{\overline{n}|}(1+i)$$

$$\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|} \quad \text{and} \quad \ddot{s}_{\overline{n}|} = s_{\overline{n+1}|} - 1$$

Example 4 An investor wishes to accumulate \$1000 in a fund at end of 12 years. To accomplish this the investor plans to make deposits at the end of each year, the final payment to be made one year prior to the end of the investment period. How large should each deposit be if the fund earns 7% effective.

Since we are interested in the accumulated value one year after the last payment, the equation of value is

$$R\ddot{s}_{\overline{11|}} = 1000$$

where R is the annual deposit. Solving for R we have

$$R = \frac{1000}{\ddot{s}_{\overline{11|}}} = \frac{1000}{1.07s_{\overline{11|}}} = \frac{1000}{(1.07)(15.7836)} = \$59.21$$

1.4. Annuity Value on Any Date

1) Present values more than one period before the first payment.

2) Accumulated values more than one period after the last payment date.

3) Current values between the first and last payment dates.

Present values more than one period before the first payment date

This type of annuity is often called a **deferred annuity**, since the payments commence only after a deferred period. In general, the present value annuity-immediate deferred for m periods with a term n periods after the deferred period is

$$v^m a_{\overline{n}|} = a_{\overline{n+m}|} - a_{\overline{m}|}.$$

Accumulated values more than one period after the last payment date

In general, the accumulated value of an n-period annuity, m periods after the last payment date, is

$$s_{\overline{n}|}(1+i)^m = s_{\overline{n+m}|} - s_{\overline{m}|}.$$

Current values between the first and last payment dates

In general, the current value of an n-period annuity immediately after the mth payment has been made $(m < n)$ is

$$a_{\overline{n}|}(1+i)^m = v^{n-m} s_{\overline{n}|} = s_{\overline{m}|} + a_{\overline{n-m}|}.$$

1.5. Perpetuities

A **perpetuity** is an annuity whose payments continue forever, i.e. the term of the annuity is not finite.

$a_{\overline{\infty}|}$ The present value of a perpetuity-immediate

$$\begin{aligned} a_{\overline{\infty}|} &= v + v^2 + v^3 + \cdots \\ &= \frac{v}{1-v} = \frac{v}{iv} \\ &= \frac{1}{i} \end{aligned}$$

provide $v < 1$, which will be the case if $i > 0$.
Alternatively we have

$$a_{\overline{\infty}|} = \lim_{n\to\infty} a_{\overline{n}|} = \lim_{n\to\infty} \frac{1-v^n}{i} = \frac{1}{i}$$

since $\lim_{n\to\infty} v^n = 0$.

By an analogous,for a perpetuity-due, we have

$$\ddot{a}_{\overline{\infty|}} = \frac{1}{d}$$

It should be noted that **accumulated values** for perpetuities do not exist, since the payments continue forever.

Example 5 A leaves an estate of \$100,000. Interest on the estate is paid to beneficiary B for the first 10 years, to beneficiary C for the second 10 years, and to charity D thereafter. Find the relative shares of B, C, and D in the estate, if it assumed the estate will earn a 7% annual effective rate of interest.

The value of B's share is

$$7000 a_{\overline{10|}} = 7000(7.0236) = \$49,165$$

to the nearest dollar.

The value of C's share is

$$7000(a_{\overline{20}|} - a_{\overline{10}|}) = 7000(10.5940 - 7.0236) = \$24,993$$

to the nearest dollar.

The value of D's share is

$$7000(a_{\overline{\infty}|} - a_{\overline{20}|}) = 7000(\frac{1}{.07} - 10.5940) = \$25,842$$

to the nearest dollar.

Note that the sum of the shares of B,C, and D is equal to \$100,000 as expected. Also note that the present value of the estate at the end of 20 years is $100,000(1.07)^{-20} = \$25,842$, to the nearest dollar, which is equal to D's share. This confirms the fact that charity D continuing to receive the interest into perpetuity or receiving the estate value in a lump-sum at the end of 20 years are equivalent in value.

1.6. Nonstandard term and Interest rates

Consider first what the symbol $a_{\overline{n+k}|}$, where n is a positive integer and $0 < k < 1$.

$$\begin{aligned} a_{\overline{n+k}|} &= \frac{1 - v^{n+k}}{i} = \frac{1 - v^n + v^n - v^{n+k}}{i} \\ &= a_{\overline{n}|} + v^{n+k}\left[\frac{(1+i)^k - 1}{i}\right]. \end{aligned}$$

It is the present of an n-period annuity-immediate of 1 per period plus a final payment at time $n+k$ of $\frac{(1+i)^k-1}{i}$.

$\frac{(1+i)^k-1}{i}$ is an irregular payment, seems rather unusual. A payment k is a good approximation to it.

In practice many courts use an annuity-certain for a person's life expectancy in measurement of economic damages in personal injury and wrongful death lawsuits.

1.7. Unknown Time

In general, problems involving unknown time will not produce exact integral answer for n

These problems could be handle along the liens of the section above in which a small payment is made during the period following the last regular payment. However it is seldom done in practice. The final smaller payment date is not convenient for either party to the transaction.

What is usually done in practice is either to make a smaller additional payment at the same time as the last regular payment, in effect making a payment larger than the regular payment (*ballon payment*), or to make a smaller payment one period after the last regular payment (*drop payment*).

Example 6 An investment of \$1000 is to be used to make payments of \$100 at the end of every year for as long as possible. If the fund earns an annual effective rate of interest of 5%, find how many regular payments can be made and find the amount of the smaller payment: (1) to be paid on the date of the last regular payment, (2) to be paid one year after the last regular payment, and (3) to be paid during the year following the last regular payment.

The equation of value is $100a_{\overline{n}|} = 1000$ or $a_{\overline{n}|} = 10$. By inspection of interest tables, we have $14 < n < 15$. Thus 14 regular payments can be made plus a small final payment.

1. The equation of value at the end of 14 year is

$$100s_{\overline{14}|} + X_1 = 1000(1.05)^{14}$$

Thus, $X_1 = 1000(1.05)^{14} - 100s_{\overline{14}|} = 1979.93 - 1959.86 = \$20.07$.

2. The equation of value at end of 15th year is

$$100\ddot{s}_{\overline{14|}} + X_2 = 1000(1.05)^{15}$$

Thus, $X_2 = 1000(1.05)^{15} - 100(s_{\overline{15|}} - 1) = 2078.93 - 2057.86 = \$21.07$. It should be noted that $20.07(1.05) = 21.07$, or that in general $X_1(1+i) = X_2$.

3. In this case the equation of value becomes

$$100a_{\overline{14+k|}} = 1000$$

where $0 < k < 1$. This can be written as

$$\frac{1 - v^{n+k}}{i} = 10 \quad \text{or} \quad v^{14+k} = 1 - 10i = .5.$$

Thus $(1.05)^{14+k} = 2$. Hence $k = .2067$. Thus, the exact final irregular payment is

$$X_3 = 100\frac{(1.05)^{.2067} - 1}{.05} = \$20.27$$

paid at time 14.2067. The exact answer obtained between the answer to 1 and 2, as we expect.

1.8. Unknown rate of Interest

There are three methods to use in determining an unknown rate of interest.

- Solve for i by **algebraic techniques**.

$$a_{\overline{n}|} = v + v^2 + \cdots + v^n$$

is an nth degree polynomial in v. If the roots of this polynomial can be determined algebraically, then i is immediately determined.
We can express $a_{\overline{n}|}$ and $\frac{1}{a_{\overline{n}|}}$ in terms of i

$$a_{\overline{n}|} = n - \frac{n(n+1)}{2!}i + \frac{n(n+1)(n+2)}{3!}i^2 + \cdots,$$

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{n}\left[1 + \frac{n+1}{2}i + \frac{n^2-1}{12}i^2 + \cdots\right]$$

- To use **linear interpolation** in the interest tables. It should be noted that the accuracy of linear interpolation will depend on how closely together the interest rates appearing in the tables are tabulated.

- **Successive approximation or iteration.** This method can be applied to all problems of this type and can produce whatever level of accuracy is required by carrying out enough iterations.

  Iteration can easily be applied when an equation of the form $i = g(i)$ exists and converges to the true value of i. Assume some starting value i_0, then generate a value i_1 and so on,

$$i_1 = g(i_0) \rightarrow i_2 = g(i_1) \ldots i_n = g(i_{n-1}) \ldots.$$

  If the iteration is convergence,then the successive $i_0, i_1, i_2, \ldots$ will converge to the true value i.

Iteration method use to find the rate of interest i.

Directly from Formula

$$i = \frac{1 - (1+i)^{-n}}{k}$$

where $k = a_{\overline{n}|}$. The rate converge of iteration formula is quite slow.

Newton-Raphson iteration

$$i_{s+1} = i_s \left[1 + \frac{1 - (1+i_s)^{-n} - k i_s}{1 - (1+i_s)^{-n-1}\{1 + i_s(n+1)\}} \right]$$

It does have a very rapid rate of convergence.

Starting value, good starting value can be obtained by linear interpolation in interest table. A more convenient method is to apply diretly an approximation formula based only on n and $a_{\overline{n}|} = k$:

$$\frac{1}{k} = \frac{1}{a_{\overline{n}|}} \approx \frac{1}{n}\left[1 + \frac{n+1}{2}i\right]$$

which gives

$$i \approx \frac{2(n-k)}{k(n+1)}.$$

Analogous results can be derived for accumulated value. Here we let $s_{\overline{n}|i} = k$.

$$i_{s+1} = i_s\left[1 + \frac{(1+i_s)^n - 1 - ki_s}{(1+i_s)^{n-1}\{1 - i_s(n-1)\} - 1}\right]$$

and starting value

$$i \approx \frac{2(k-n)}{k(n-1)}.$$

Example 7 At what of interest, convertible quarterly, is \$16,000 the present value of \$1000 paid at the end of every quarter for five years ?
Let $j = i^{(4)}/4$, so that the equation of value becomes

$$1000a_{\overline{20}|j} = 16,000 \quad \text{or} \quad a_{\overline{20}|j} = 16.$$

We will first illustrate interpolation in the interest tables. Define

$$f(j) = a_{\overline{20}|j} - 16.$$

We seek to find j, such that $f(j) = 0$. By inspection of the interest tables,

$$f(.0200) = 16.3514 - 16 = .3514 \quad \text{and} \quad f(.0250) = 15.5892 - 16 = -.4108.$$

Now performing a linear interpolation

$$j = .0200 + 0.005\frac{.3514 + 0}{.3514 + .4108} = .0223$$

which gives $i^{(4)} = 4(.0223) = .0892$ or 8.92%.

We will next illustrate iteration. For a starting value we could use the value derived by linear interpolation, i.e. $j_0 = .0223$. However, we will instead apply formula above to obtain

$$\frac{2(20-16)}{(16)(20)} = .0238.$$

We first illustrate the first iteration method. We obtain the following results

$$\begin{array}{cccc} j_0 = .02380 & j_4 = .02283 & j_8 = .02283 & j_{12} = .02234 \\ j_1 = .02345 & j_5 = .02270 & j_9 = .02243 & j_{13} = .02233 \\ j_2 = .02319 & j_6 = .02261 & j_{10} = .02239 & j_{14} = .02231 \\ j_3 = .02298 & j_7 = .02253 & j_{11} = .02237 & j_{15} = .02230 \end{array}$$

We stop after 15 cycles, since the iteration is converging so slowly. Here we just want to demonstrate the care which must be taken in practice to use iteration methods with a reasonable rate of convergence.

We now try the Newton-Raphson iteration method. We obtain the following results

$$j_0 = .02380 \quad j_1 = .0222459 \quad j_2 = .0222623 \quad j_3 = .0222623$$

We have achieved seven decimal places of accuracy after only two iterations!. The slow ad hoc method above could not even achieve five decimal places of accuracy after 15 iterations. The power of the Newton-Raphson method is indeed evident.

The correct answer to six decimal places is thus

$$i^{(4)} = 4(.0222623) = .089049 \quad \text{or} \quad 8.9049\%.$$

1.9. Varying Interest

Two patterns of variation could be involved. The first pattern would be for i_k to be the applicable rate for period k regardless of when the payment is made. In this case the present value becomes

$$\begin{aligned} a_{\overline{n}|} &= (1+i_1)^{-1} + (1+i_1)^{-1}(1+i_2)^{-1} \\ &\quad + \ldots + (1+i_1)^{-1}(1+i_2)^{-1} \cdots (1+i_n)^{-1} \\ &= \sum_{t=1}^{n} \prod_{s=1}^{t} (1+i_s)^{-1} \end{aligned}$$

The second patter would be compute present values using rate i_k for the payment made at time k over all k periods. In this case the present value becomes

$$\begin{aligned} a_{\overline{n}|} &= (1+i_1)^{-1} + (1+i_2)^{-2} + \ldots + (1+i_n)^{-n} \\ &= \sum_{t=1}^{n} (1+i_t)^{-t}. \end{aligned}$$

First pattern, for accumulated values, we have

$$\begin{aligned}
\ddot{s}_{\overline{n}|} &= (1+i_n) + (1+i_n)(1+i_{n-1}) \\
&\quad + \ldots + (1+i_n)(1+i_{n-1})\cdots(1+i_1) \\
&= \sum_{t=1}^{n} \prod_{s=1}^{t} (1+i_{n-s+1})
\end{aligned}$$

For second pattern, we have

$$\begin{aligned}
\ddot{s}_{\overline{n}|} &= (1+i_n) + (1+i_{n-1})^2 + \ldots + (1+i_1)^n \\
&= \sum_{t=1}^{n} (1+i_{n-t+1})^t.
\end{aligned}$$

Accumulated values of the annuity-immediate can be obtained from accumulated values of the annuity-due by using formula $s_{\overline{n+1}|} = \ddot{s}_{\overline{n}|} + 1$.

Example 8 Find the accumulated value of a 10-year annuity-immediate of \$100 per year if the effective rate of interest is 5% for the first 6 years and 4% for the last 4 years.

The answer is

$$
\begin{aligned}
& 100\left[s_{\overline{6}|.05}(1.04)^4 + s_{\overline{4}|.04}\right] \\
= \quad & 100[(6.8019)(1.16986) + 4.2465] = 1220.38.
\end{aligned}
$$

Example 9 Rework Example above if the first 6 payments are invested at an effective rate of interest 5% and if the final 4 payments are invested at 4%

The answer is

$$
\begin{aligned}
& 100\left[s_{\overline{6}|.05}(1.05)^4 + s_{\overline{4}|.04}\right] \\
= \quad & 100[(6.8019)(1.21551) + 4.2465] = 1251.43.
\end{aligned}
$$

1.10. Annuities not involving compound interest

The valuation of annuities not involving compound interest is full of pitfalls and requires careful analysis and interpretation to obtain reasonable results.

Unfortunately, it is not always possible to avoid this subject.

The subject of annuity values not involving compound interest has appeared in various forms in the literature. You should be exposed to some of the results which have appeared and ambiguities.

Generalized version for $a_{\overline{n}|}$

$$a_{\overline{n}|} = \sum_{t=1}^{n} \frac{1}{a(t)}$$

If we assume 1 invested at time t, where $t = 1, 2, \ldots, n-1$ will accumulate to $\frac{a(n)}{a(t)}$ at time n, then we have

$$s_{\overline{n}|} = \sum_{t=0}^{n-1} \frac{a(n)}{a(t)} = a(n) \sum_{t=1}^{n} \frac{1}{a(t)}. \tag{1}$$

The above formula does not produce correct results in all case. For example, suppose we wish to find the accumulated value of an n-period annuity-immediate in which each payment is invested at a simple interest from the day of payment until the end of the n periods. The accumulated value of such an annuity would be

$$1 + (1+i) + (1+2i) + \cdots + [1 + (n-1)i]$$

.

It lead to Generalized version for $s_{\overline{n}|}$

$$s_{\overline{n}|} = \sum_{t=0}^{n-1} a(t) \tag{2}$$

which will produce the correct answer for the above simple interest example.

By the formula above, we can find another expression for $a_{\overline{n}|}$

$$a_{\overline{n}|} = \sum_{t=0}^{n-1} \frac{a(t)}{a(n)} = \frac{1}{a(n)} \sum_{t=0}^{n-1} a(t).$$

It is not surprising that formulas above for $a_{\overline{n}|}$ produce different answers unless compound interest is involved.

Example 10 Compare the value of $s_{\overline{6}|.1}$ at 10% interest (1) assuming compound interest, (2) using formula (1), and (3) using formula (2).

1. Using compound interest at 10%
$$s_{\overline{6}|} = 7.72.$$

2. From formula (1)
$$s_{\overline{6}|} = 1.6\left[\frac{1}{1.1} + \frac{1}{1.2} + \cdots + \frac{1}{1.6}\right] = 7.23.$$

3. From formula (2)
$$s_{\overline{6}|} = 1 + 1.1 + 1.2 + \cdots + 1.5 = 7.5.$$

Answer (3) involves simple interest being earned on each payment from the date of deposit to the end of the six-year period of investment. Answer 1 is larger than either of the other two, verifying the larger accumulation with compound interest.

2^*. More General Annuities

In Chapter 3 we discussed annuities for which the payment period and the interest conversion period coincide, and for which the payments are all same.

Annuities for which payments are made more or less frequently than interest is convertible and annuities with varying payments will be considered.

2.1. Annuities payable at a different frequency than interest is convertible

- If the objective is to compute the numerical value of an annuity, a two-step procedure can be followed:

  1. Find the rate of interest, convertible at the same frequency as payments are made, which is equivalent to the given rate of interest.
  2. Using this new rate of interest, find the value of the annuity using the technique discussed in Chapter 3.

- Develop algebraic expressions for which such annuities in terms of annuity symbols already defined in Chapter 3, with adjustment factors sometimes being required.

Example 1 Find the accumulated value at end of four years of an investment fund in which \$100 is deposited at the beginning of each quarter for the first two years and \$200 is deposited at the beginning of each quarter for the second two years, if the fund earned is 12% convertible monthly.

Let j be the equivalent rate of interest per quarter, which is the payment period. We have

$$j = (1 + .12/12)^3 - 1 = .030301.$$

The value of the annuity is

$$100(\ddot{s}_{\overline{16}|j} + b\ddot{s}_{\overline{8}|j}) = 100(20.8170 + 9.1716) = \$2999$$

to the nearest dollar.

Example 2 A loan of \$3000 is to be repaid with quarterly installments at end quarter for five years. If the rate of interest charged on the loan is 10% convertible semiannually, find the amount of each monthly payment

Let j be the equivalent rate of interest per quarter, which is the payment period. we have

$$j = (1.05)^{1/2} - 1 = .024695.$$

Let the quarterly payment be R. Then the equation of value is $Ra_{\overline{20}|j} = 3000$, so that

$$R = \frac{3000}{a_{\overline{20}|j}} = \frac{3000}{15.6342} = \$191.89$$

2.2. Further analysis of Annuities payable less frequently than interest is convertible

- Annuity-immediate

$$
\begin{aligned}
v^k + v^{2k} + \cdots + v^{\frac{n}{k}\cdot k} &= \frac{v^k - v^{n+k}}{1 - v^k} \\
&= \frac{1 - v^n}{(1+i)^{-1}} = \frac{a_{\overline{n}|}}{s_{\overline{k}|}}
\end{aligned}
$$

$$
\frac{a_{\overline{n}|}}{s_{\overline{k}|}}(1+i)^n = \frac{s_{\overline{n}|}}{s_{\overline{k}|}}
$$

- Annuity-due

$$
1 + v^k + v^{2k} + \cdots + v^{n-k} = \frac{1 - v^n}{1 - v^k} = \frac{a_{\overline{n}|}}{a_{\overline{k}|}}
$$

$$
\frac{a_{\overline{n}|}}{a_{\overline{k}|}}(1+i)^n = \frac{s_{\overline{n}|}}{a_{\overline{k}|}}
$$

- Other considerations

1. Perpetuity payable is less frequent than interest
   The present value of perpetuity-immediate is $\frac{1}{is_{\overline{k|}}}$.

   The present value of perpetuity-due is $\frac{1}{ia_{\overline{k|}}}$.
2. To find the value of a series of payments at a given force of δ. Replace v^{tk} with $e^{-\delta tk}$ and $(1+i)^{tk}$ with $e^{\delta tk}$.
3. Each payment period does not contain an integral number of interest conversion periods.

Example 3 An investment of \$1000 is used to make payments of \$100 at the end of each year for as long as possible with a smaller final payment to be made at the time of the last regular payment. If interest is 7% convertible semiannually, find the number of payments and the amount of the total final payment

The equation of value is

$$100\frac{a_{\overline{n}|.035}}{s_{\overline{2}|.035}} = 1000$$

By inspection of the interest tables , we have $36 < n < 37$. Thus,18 regular payments and a small final payment can be made. Let the smaller additional payment at the time of the final regular payment be denoted by R. Then an equation of value at the end of 18 years is

$$R + 100\frac{s_{\overline{36}|.035}}{s_{\overline{2}|.035}} = 1000(1.035)^{36}$$

or

$$R = 1000(3.45027) - 100\frac{70.0076}{2.0350} = \$10.09.$$

The total final payment would thus be \$110.09.

2.3. Further analysis of Annuities Payable more frequently than Interest is convertible

Annuity-immediate

m: the number of payment periods in one interest conversion period
n: the term of the annuity measured in interest conversion period
i : the interest rate per interest conversion period.
$a_{\overline{n}|}^{(m)}$: the present value of an annuity which pays $\frac{1}{m}$ at the end of each mth of an interest conversion period for a total of n interest conversion periods

$$\begin{aligned} a_{\overline{n}|}^{(m)} &= \frac{1}{m}\left[v^{\frac{1}{m}} + v^{\frac{2}{m}} + \cdots + v^{n-\frac{1}{m}} + v^{n}\right] \\ &= \frac{1}{m}\left[\frac{v^{\frac{1}{m}} - v^{n+\frac{1}{m}}}{1 - v^{\frac{1}{m}}}\right] \\ &= \frac{1-v^n}{m((1+i)^{\frac{1}{m}-1})} = \frac{1-v^n}{i^{(m)}}. \end{aligned}$$

$$s_{\overline{n}|}^{(m)} = a_{\overline{n}|}^{(m)}(1+i)^n = \frac{(1+i)^n - 1}{i^{(m)}}.$$

Write $a_{\overline{n}|}^{(m)}$ and $s_{\overline{n}|}^{(m)}$ in terms of $a_{\overline{n}|}$ and $s_{\overline{n}|}$ with an adjustment factor

$$a_{\overline{n}|}^{(m)} = \frac{i}{i^{(m)}} a_{\overline{n}|}$$

$$s_{\overline{n}|}^{(m)} = \frac{i}{i^{(m)}} s_{\overline{n}|}$$

Annuity-due

$\ddot{a}_{\overline{n}|}^{(m)}$: the present value of an annuity which pays $\frac{1}{m}$ at the beginning of each mth of an interest conversion period for a total of n interest conversion periods.

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{1 - v^n}{d^{(m)}}$$

$$\ddot{s}_{\overline{n}|}^{(m)} = \ddot{a}_{\overline{n}|}^{(m)} (1+i)^n = \frac{(1+i)^n - 1}{d^{(m)}}$$

Write $\ddot{a}^{(m)}_{\overline{n}|}$ and $\ddot{s}^{(m)}_{\overline{n}|}$ in terms of $a_{\overline{n}|}$ and $s_{\overline{n}|}$ with an adjustment factor

$$\ddot{a}^{(m)}_{\overline{n}|} = \frac{i}{d^{(m)}} a_{\overline{n}|}$$

$$\ddot{s}^{(m)}_{\overline{n}|} = \frac{i}{d^{(m)}} s_{\overline{n}|}$$

$$\ddot{a}^{(m)}_{\overline{n}|} = (1+i)^{\frac{1}{m}} a^{(m)}_{\overline{n}|} = \left[1 + \frac{i^{(m)}}{m}\right] \frac{i}{i^{(m)}} a_{\overline{n}|} = \left[\frac{i}{i^{(m)}} + \frac{i}{m}\right] a_{\overline{n}|}$$

$$\ddot{s}^{(m)}_{\overline{n}|} = \left[\frac{i}{i^{(m)}} + \frac{i}{m}\right] s_{\overline{n}|}$$

Other Consider

- A perpetuity payable more frequently than interest is convertible

$$a_{\overline{\infty}|}^{(m)} = \frac{1}{i^{(m)}}, \quad \text{and} \quad \ddot{a}_{\overline{\infty}|}^{(m)} = \frac{1}{d^{(m)}}$$

- Each interest conversion period does not contain an integral number of payment periods.(i.e. $m > 1$, but m is not integral).

- Proper coefficient for annuities payable $m$thly.
  The proper coefficient is the amount paid during one interest conversion period (*periodic rent, annual rent of annuity*).

Example 4 Payments of \$400 per month are made over a ten-year period. Find expression for: (1) the present value of these payments two years prior to the first payment, and (2) the accumulated value three years after the final payment. Use symbols based on an effective rate of interest.

1) The answer is

$$4800v^2\ddot{a}^{12}_{\overline{10}|} = 4800(\ddot{a}^{12}_{\overline{12}|} - \ddot{a}^{12}_{\overline{2}|}).$$

2) The answer is

$$4800s^{12}_{\overline{10}|}(1+i)^3 = 4800(s^{12}_{\overline{13}|} - s^{12}_{\overline{3}|}).$$

2.4. Continuous Annuities

$\bar{a}_{\overline{n}|}$: the present value of an annuity payable continuously for n interest conversion periods, such that the total amount paid during each interest conversion period is 1.

$$\bar{a}_{\overline{n}|} = \int_0^n v^t dt = \frac{v^t}{\log_e v}\Big|_0^n = \frac{1-v^n}{\delta}.$$

$$\bar{a}_{\overline{n}|} = \lim_{m\to\infty} a_{\overline{n}|}^{(m)} = \lim_{m\to\infty} \frac{1-v^n}{i^{(m)}} = \frac{1-v^n}{\delta}.$$

$$\bar{a}_{\overline{n}|} = \lim_{m\to\infty} \ddot{a}_{\overline{n}|}^{(m)} = \lim_{m\to\infty} \frac{1-v^n}{d^{(m)}} = \frac{1-v^n}{\delta}.$$

$$\bar{a}_{\overline{n}|} = \frac{i}{\delta} a_{\overline{n}|} = \bar{s}_{\overline{1}|} a_{\overline{n}|}.$$

$$\bar{s}_{\overline{n}|} = \int_0^n (1+i)^t dt = \frac{(1+i)^n - 1}{\delta} = \frac{i}{\delta} s_{\overline{n}|} = \bar{s}_{\overline{1}|} s_{\overline{n}|} = \lim_{m\to\infty} s_{\overline{n}|}^{(m)} = \lim_{m\to\infty} \ddot{s}_{\overline{n}|}^{(m)}$$

2.5. Basic Varying Annuities

We now consider annuities with varying payments. In this section, it will be assumed that the payment period and interest conversion period are equal and coincide.

- Payments varying in arithmetic progression.
- Payments varying in geometric progession.
- Other payment patterns.

Payment varying in arithmetic progression

A : the present value of the annuity.
P : the beginning payment. Q: increasing payment per period thereafter.

$$\begin{aligned} A &= Pv + (P+Q)v^2 + (P+2Q)v^3 + \cdots \\ &\quad +[(p+(n-2)Q]v^{n-1} + [p+(n-1)Q]v^n. \\ &= P\frac{1-v^n}{i} + Q\frac{a_{\overline{n}|} - nv^n}{i} \\ &= Pa_{\overline{n}|} + Q\frac{a_{\overline{n}|} - nv^n}{i}. \end{aligned}$$

The accumulated value:

$$Ps_{\overline{n}|} + Q\frac{s_{\overline{n}|} - n}{i}.$$

Special cases

- Increasing annuity $P = 1$ and $Q = 1$

$$(Ia)_{\overline{n}|} = a_{\overline{n}|} + \frac{a_{\overline{n}|} - nv^n}{i} = \frac{\ddot{a}_{\overline{n+1}|} - (n+1)v^n}{i} = \frac{\ddot{a}_{\overline{n}|} - (n)v^n}{i}$$

$$(Is)_{\overline{n}|} = (Ia)_{\overline{n}|}(1+i)^n = \frac{\ddot{s}_{\overline{n}|} - n}{i} = \frac{\ddot{s}_{\overline{n+1}|} - (n+1)}{i}$$

- Decreasing annuity $P = n$ and $Q = -1$

$$(Da)_{\overline{n}|} = na_{\overline{n}|} - \frac{a_{\overline{n}|} - nv^n}{i} = \frac{n - a_{\overline{n}|}}{i}$$

$$(Ds)_{\overline{n}|} = (Da)_{\overline{n}|}(1+i)^n = \frac{n(1+i)^n - s_{\overline{n}|}}{i}.$$

- General present value form for a perpetuity

$$\frac{P}{i} + \frac{Q}{i^2}$$

Payment varying in geometric progression

For such situation, the annuities can be readily handled by directly expressing the annuity value as a series with each payment multiplied by its associated present or accumulated value.

Consider an annuity-immediate with a term of n periods in which the first payment is 1 and successive payments increasing in geometric progression with common ratio (1+k). The present value is

$$\begin{aligned} v + v^2(1+k) + \cdots + v^n(1+k)^{n-1} &= v\left[\frac{1-\left(\frac{1+k}{1+i}\right)^n}{1-\left(\frac{1+k}{1+i}\right)}\right] \\ &= \frac{1-\left(\frac{1+k}{1+i}\right)^n}{i-k}. \end{aligned}$$

Other payment patterns

- Handled from first principles. The annuity value can be computed by finding the present value or accumulated value of each payment separately and summing the results.

- Distinguish the term "varying annuity" from the term "variable annuity".

More General Varying Annuities

- Payments are made less frequently than interest is convertible.

- Payments are made more frequently than interest is convertible.

    - The rate of payment is constant during each interest conversion period.
    - The rate of payment changes with each payment period.

- Payment vary in geometric progression and in which the payment period and the interest conversion period differ.

- Continuous varying annuities.