Chapter 2

Random Variable

CLO2	Define single random variables in terms of their PDF and CDF, and calculate moments such as the mean and variance.

1. Introduction

- In Chapter 1, we introduced the concept of event to describe the characteristics of outcomes of an experiment.
- Events allowed us more flexibility in determining the proprieties of the experiments better than considering the outcomes themselves.
- In this chapter, we introduce the concept of random variable, which allows us to define events in a more consistent way.
- In this chapter, we present some important operations that can be performed on a random variable.
- Particularly, this chapter will focus on the **concept of expectation and variance**.

2. The random variable concept

- A random variable X is defined as a real function that maps the elements of sample space *S* to real numbers (function that maps all elements of the sample space into points on the real line).

$$X: S \longrightarrow \mathbb{R}$$

- A random variable is denoted by a capital letter (such as: X, Y, Z) and any particular value of the random variable by a lowercase letter (such as: x, y, z).
- We assign to s (every element of S) a real number X(s) according to some rule and call X(s) a random variable.

Example 2.1:

An experiment consists of flipping a coin and rolling a die.

Let the random variable *X* chosen such that:

A coin head (*H*) corresponds to positive values of *X* equal to the die number

A coin tail (*T*) corresponds to negative values of *X* equal to twice the die number.

Plot the mapping of *S* into *X*.

Solution 2.1:

The random variable *X* maps the samples space of 12 elements into 12 values of *X* from -12 to 6 as shown in Figure 1.

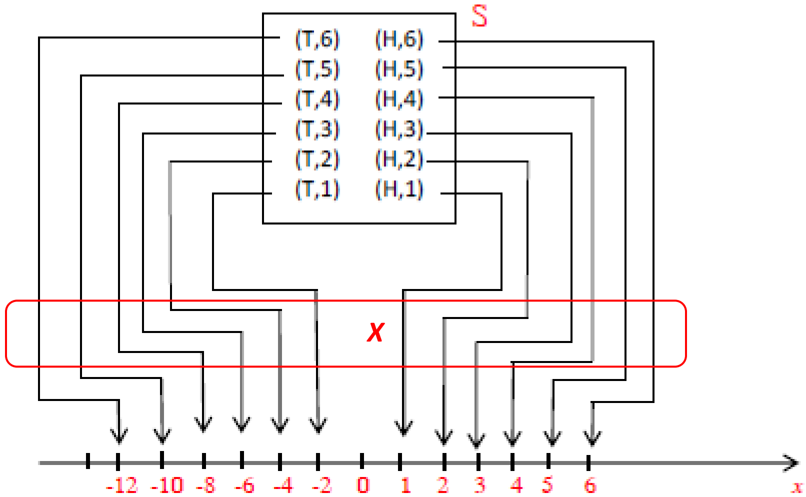


Figure 1. A random variable mapping of a sample space.

- **Discrete random variable:** If a random variable X can take only a particular finite or counting infinite set of values $x_1, x_2, \ldots, x_N$, then X is said to be a discrete random variable.

- **Continuous random variable:** A continuous random variable is one having a continuous range of values.

3. Distribution function

- If we define $P(X \leq x)$ as the probability of the event $X \leq x$ then the ***cumulative probability distribution function*** $\boldsymbol{F_X(x)}$ or often called ***distribution function*** of X is defined as:

$$F_X(x) = P(X \leq x) \quad for \quad -\infty < x < \infty \qquad (1)$$

Probability mass function

- The argument x is any real number ranging from $-\infty$ to ∞.

- **Proprieties:**

  1) $F_X(-\infty) = 0$
  2) $F_X(\infty) = 1$

     ($since\ F_X$ is a probability, the value of the distribution function is always between 0 and 1).

  3) $0 \leq F_X(x) \leq 1$
  4) $F_X(x_1) \leq F_X(x_2)$ if $x_1 < x_2$ (event $\{X \leq x_1\}$ is contained in the event $\{X \leq x_2\}$, monotically increasing function)
  5) $P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$
  6) $F_X(x^+) = F_X(x)$, where $x^+ = x + \varepsilon$ and $\varepsilon \to 0$ (Continuous from the right)

- For a discrete random variable *X*, the distribution function $F_X(x)$ must have a "*stairstep form*" such as shown in Figure 2.

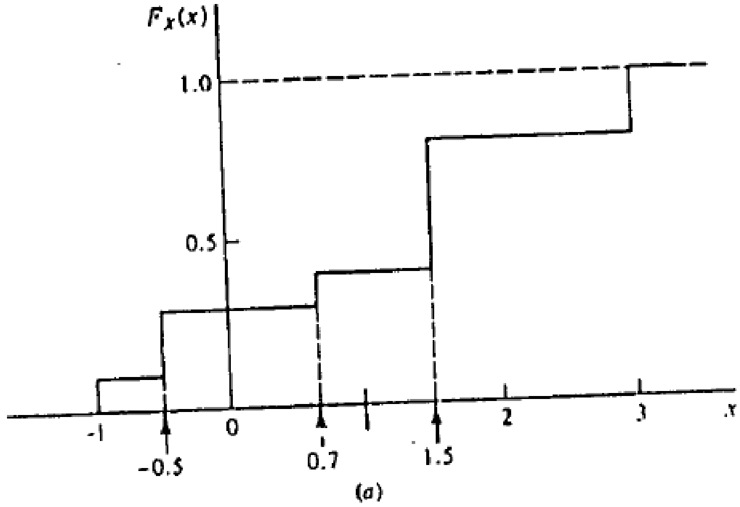


Figure 2. Example of a distribution function of a discrete random variable.

- The amplitude of a step equals to the probability of occurrence of the value X where the step occurs, we can write:

$P(X = x_i)$

$$F_X(x) = \sum_{i=1}^{N} P(x_i) \cdot u(x - x_i) \quad (2)$$

Unit step function: $u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$

4. Density function

- The **probability density function (pdf),** denoted by $\boldsymbol{f_X(x)}$ is defined as the derivative of the distribution function:

$$F_X(x) = \int_{-\infty}^{x} f_X(\theta) d\theta$$

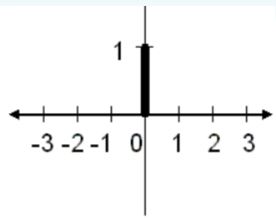
$$f_X(x) = \frac{dF_X(x)}{dx} \quad (3)$$

- $f_X(x)$ is often called the density function of the random variable *X*.

- For a discrete random variable, this density function is given by:

$$f_X(x) = \sum_{i=1}^{N} P(x_i)\delta(x - x_i) \qquad (4)$$

δ Unit impulse function: $\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & otherwise \end{cases}$



- **Proprieties:**
  - ✓ $f_X(x) \geq 0$ for all x
  - ✓ $F_X(x) = \int_{-\infty}^{x} f_x(\theta)d\theta$
  - ✓ $\int_{-\infty}^{\infty} f_X(x)dx = F_X(\infty) - F_X(-\infty) = 1$
  - ✓ $P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(\theta)d\theta$

Example 2.2:

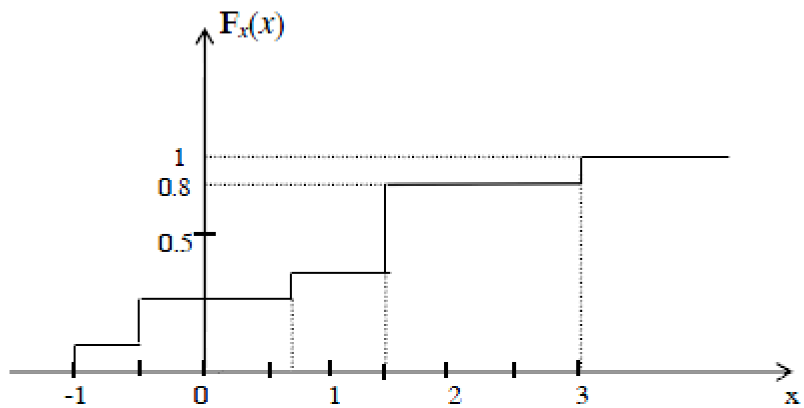
Let *X* be a random variable with discrete values in the set {-1, -0.5, 0.7, 1.5, 3}. The corresponding probabilities are assumed to be {0.1, 0.2, 0.1, 0.4, 0.2}.

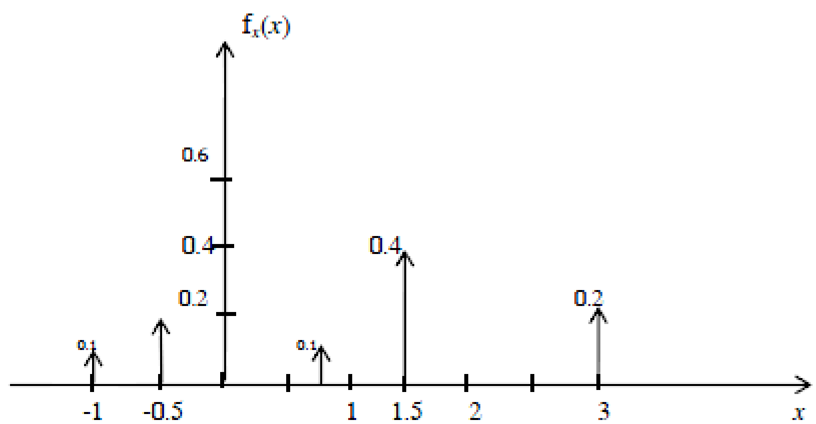
a) Plot $F_X(x), and\ f_X(x)$

b) Find $P(x < -1)$, $P(-1 < x \leq -0.5)$

Solution 2.2:

a)





b) P(X<-1) = 0 because there are no sample space points in the set {X<-1}. Only when X=-1 do we obtain one outcome and we have immediate jump in probability of 0.1 in $F_X(x)$. For -1<x<-0.5 there are no additional space points so $F_X(x)$ remains constant at the value 0.1.

$$P(-1 < X \leq -05) = F_X(-0.5) - F_X(-1) = 0.3 - 0.1 = 0.2$$

Example 3:

Find the constant *c* such that the function:

$$f_X(x) = \begin{cases} c \cdot x & 0 \leq x \leq 3 \\ 0 & otherwise \end{cases}$$

is a valid probability density function (*pdf*)

Compute $P(1 < x \leq 2)$

Find the cumulative distribution function $F_X(x)$

Solution:

5. Examples of distributions

Discrete random variables	**Continuous random variables**
• Binominal distribution • Poisson distribution	• Gaussian (Normal) distribution • Uniform distribution • Exponential distribution • Rayleigh distribution

The Gaussian distribution

- The Gaussian or normal distribution is on the important distributions as it describes many phenomena.
- A random variable *X* is called Gaussian or normal if its density function has the form:

$$f_X(x) = \frac{1}{\sqrt{2\pi{\sigma_x}^2}} e^{-\frac{(x-a)^2}{2{\sigma_x}^2}} \tag{5}$$

$\sigma_x > 0$ and a are, respectively the mean and the standard deviation of *X* which measures the width of the function.

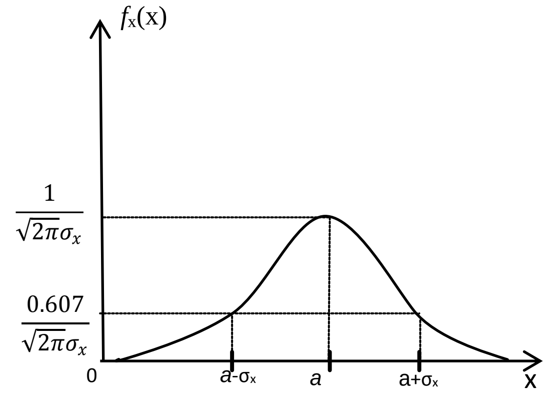


Figure 3. Gaussian density function

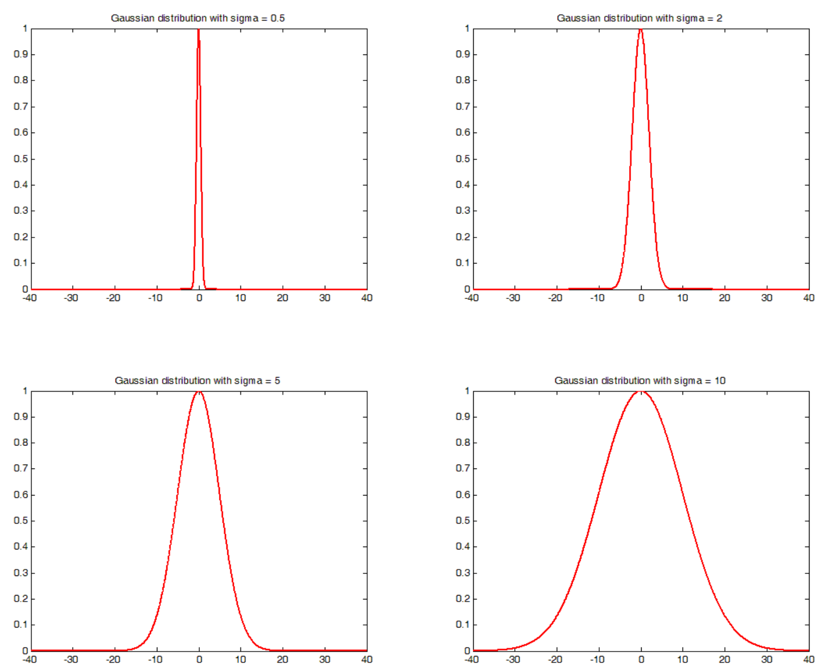
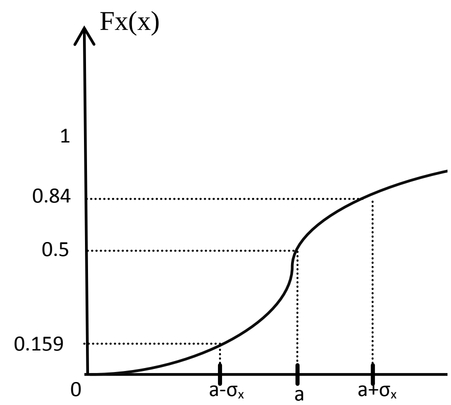


Figure 4. Gaussian density function with a =0 and different values of σ_x

- The distribution function is:

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^{x} e^{\frac{-(\theta-a)^2}{2\sigma_x^2}} d\theta \qquad (5)$$

This integral has no closed form solution and must be solved by numerical methods.



- To make the results of $F_X(x)$ available for any values of x, a, σ_x, we define a standard normal distribution with mean a = 0 and standard deviation $\sigma_x = 1$, denoted N(0,1):

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad (6)$$

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{\beta^2}{2}} d\beta \qquad (7)$$

- Then, we use the following relation:

$$F_Z(\mathrm{z}) = F_X\left(\frac{x-\mathrm{a}}{\sigma_x}\right) \qquad (8)$$

- To extract the corresponding values from an integration table developed for N(0,1).

Example 4:

Find the probability of the event {X ≤ 5.5} for a Gaussian random variable with *a*=3 and $\sigma_x = 2$

Solution:

$$P\{X \leq 5.5\} = F_Z(5.5) = F_X(\frac{5.5-3}{2}) = F_X(1.25)$$

Using the table, we have: $P\{X \leq 5.5\} = F_X(1.25) = 0.8944$

Example 5:

In example 4, find *P*{*X* > 5.5}

Solution:

$$P\{X > 5.5\} = 1 - P\{X \leq 5.5\}$$

$$= 1 - F(1.25) = 0.1056$$

6. Other distributions and density examples

The Binomial distribution

- The binomial density can be applied to the Bernoulli trial experiment which has two possible outcomes on a given trial.
- The density function $f_x(x)$ is given by:

$$f_X(x) = \sum_{k=0}^{N} \binom{N}{k} p^K (1-p)^{N-k} \delta(x-k) \quad (9)$$

Where $\binom{N}{k} = \frac{N!}{(N-k)!k!}$ and $\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$

- Note that this is a discrete r.v.
- The Binomial distribution $F_X(x)$ is:

$$F_X(x) = \int_{-\infty}^{x} \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \delta(x-k) \quad (10)$$

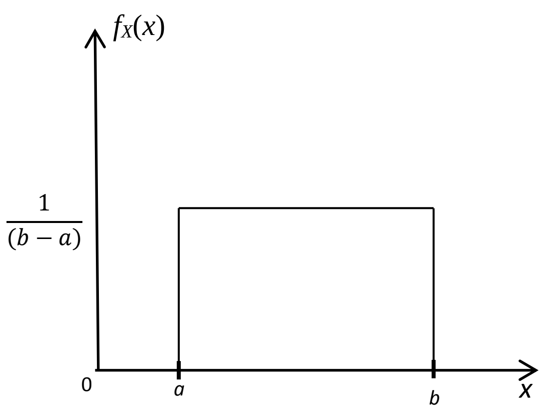
$$= \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} u(x-k)$$

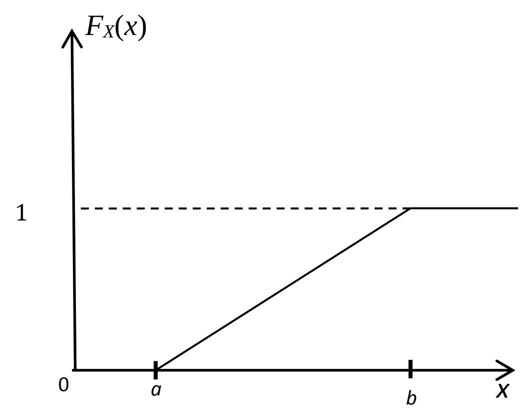
The Uniform distribution

- The density and distribution functions of the uniform distribution are given by:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & elsewhere \end{cases} \quad (11)$$

$$F_X(x) = \begin{cases} 0 & x < a \\ \frac{(x-a)}{(b-a)} & a \leq x < b \\ 1 & x \geq b \end{cases} \quad (12)$$





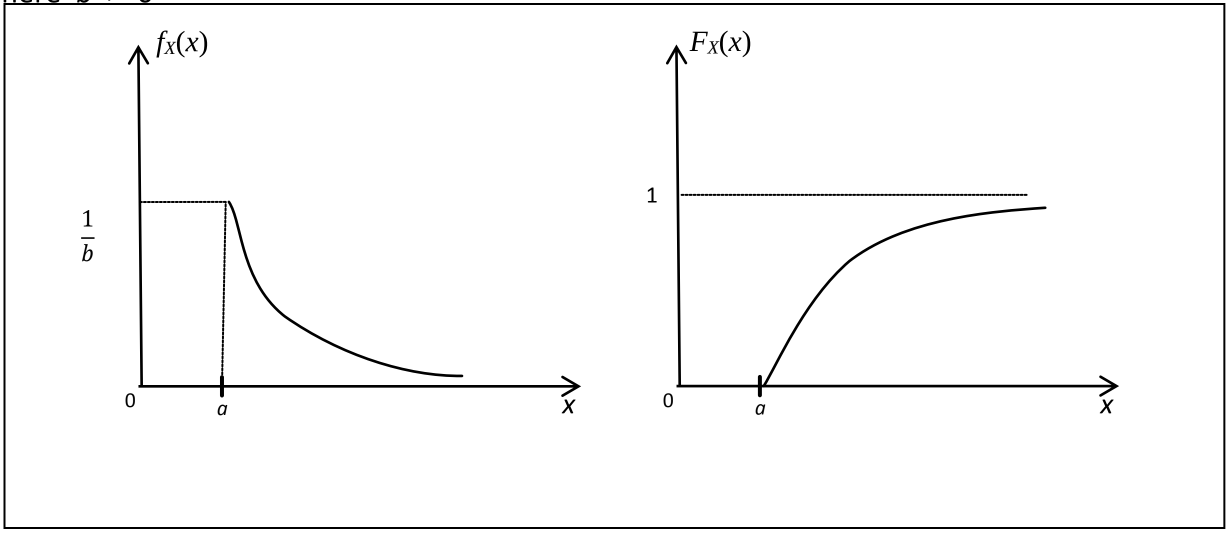
The Exponential distribution

- The density and distribution functions of the exponential distribution are given by:

$$f_X(x) = \begin{cases} \frac{1}{b} e^{\frac{-(x-a)}{b}} & x \geq a \\ 0 & x < a \end{cases} \quad (13)$$

$$F_X(x) \begin{cases} 1 - e^{\frac{-(x-a)}{b}} & x \geq a \\ 0 & x < a \end{cases} \quad (14)$$

where $b > 0$



7. Expectation

- Expectation is an important concept in probability and statistics. It is called also expected value, or mean value or statistical average of a random variable.
- The expected value of a random variable X is denoted by $E[X]$ or $\bar{X}$
- If X is a continuous random variable with probability density function $f_X(x)$, then:

$$E[X] = \int_{-\infty}^{+\infty} x f_X(x) dx \quad (15)$$

- If X is a discrete random variable having values $x_1, x_2, \dots, x_N$, that occurs with probabilities $P(x_i)$, we have

$$f_X(x) = \sum_{i=1}^{N} P(x_i)\delta(x - x_i) \quad (16)$$

Then the expected value $E[X]$ will be given by:

$$E[X] = \sum_{i=1}^{N} x_i P(x_i) \quad (17)$$

Example 3.1: find E[x] for the exponential r.v.:

$$f_X(x) = \begin{cases} \frac{1}{b} e^{\frac{-(x-a)}{b}} & x \geq a \\ 0 & x < a \end{cases}$$

Solu: $E[x] = \int_{-\infty}^{\infty} x\, f(x) dx = \int_{a}^{\infty} \frac{x}{b} e^{-\frac{(x-a)}{b}} dx = \frac{e^{a/b}}{b} \int_{a}^{\infty} x\, e^{\frac{-x}{b}} dx$

From integration table we have: $\int x e^{cx} dx = e^{cx}[\frac{x}{c} - \frac{1}{c^2}]$

Here $c = -\frac{1}{b} \Rightarrow E[X] = \frac{e^{a/b}}{b}\left[e^{-\frac{x}{b}}(-bx - b^2)\right]_a^{\infty}$

$$= \frac{e^{a/b}}{b} \; [e^{-\infty}(-\infty) - e^{-a/b}(-ab - b^2)]$$

$$= \frac{e^{a/b}.\ e^{-a/b}(ab+b^2)}{b} = a+b$$

Example 3.2: find the expected value of the points on the top face of tossing a fair die experiment?

Solu: X={1,2,3,4,5,6} and $P(x_i)=\frac{1}{6}$ for i=1,….6 since the die is fair.

So, $E[X] = \sum_{i=1}^{6} x_i\, P(x_i) = 1.\frac{1}{6} + 2.\frac{1}{6} + 3.\frac{1}{6} + 4.\frac{1}{6} + 5.\frac{1}{6} + 6.\frac{1}{6} = 3.5$

7.1 Expected value of a function of a random variable

- Let be X a random variable then the function $g(X)$ is also a random variable, and its expected value $E[g(X)]$ is given by

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx \quad (18)$$

- If X is a discrete random variable then

$$E[g(X)] = \sum_{i=1}^{N} g(x_i)P(x_i) \tag{19}$$

<u>Example 3.3</u>: A random voltage has $f_X(x) = \begin{cases} \frac{2}{5}x\, e^{\frac{-x^2}{5}} & x \geq 0 \\ 0 & x < 0 \end{cases}$

The voltage is applied to a device that generates a voltage $Y = g(x) = X^2$, which is equal to the power in 1Ω resistor. Find the average power in X?

<u>Solu</u>: Power in $X = E[g(x)] = E[X^2] = \int_0^\infty x^2 \frac{2x}{5} e^{-x^2/5} dx = \frac{2}{5}\int_0^\infty x^3 e^{-x^2/5} dx$

Let $\beta = \frac{x^2}{5}$, $d\beta = \frac{2x}{5} dx$ and $\int xe^{cx}dx = e^{cx}[\frac{x}{c} - \frac{1}{c^2}]$

Power in $X = \int_0^\infty x^2 e^{\frac{-x^2}{5}} . \frac{2x}{5} dx = \int_0^\infty 5\beta\, e^{-\beta} d\beta = 5[e^{-\beta}\left(\frac{\beta}{-1} - \frac{1}{1}\right)]_0^\infty = 5[0 - (0-1) = 5\, Watts$

8. Moments

- An immediate application of the expected value of a function $g(\cdot)$ of a random variable X is in calculating moments.
- Two types of moments are of particular interest, those **about the origin** and those **about the mean.**

8.1 Moments about the origin

- The function $g(X) = X^n, n = 0, 1, 2, \ldots$ gives the moments of the random variable X.
- Let us denote the n^{th} moment about the origin by m_n then:

$$m_n = E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) dx \tag{20}$$

$m_0 = 1$ is the area of the function $f_x(x)$.

$m_1 = E[X]$ is the expected value of X.

$m_2 = E[X^2]$ is the second moment of X.

8.2 Moments about the mean (Central moments)

- Moments about the mean value of X are called central moments and are given the symbol μ_n.
- They are defined as the expected value of the function

$$g(X) = (X - E[X])^n, n = 0,1, \dots. \tag{21}$$

Which is

$$\mu_n = E[(X - E(X))^n] = \int_{-\infty}^{\infty} (x - E[X])^n f_X(x) dx \tag{22}$$

Notes:

$u_0 = 1$, the area of $f_X(x)$

$u_1 = \int_{-\infty}^{\infty} x f_X(x) dx - E[X] \int_{-\infty}^{\infty} f_X(x) dx = 0$

8.2.1 Variance

The variance is an important statistic and it measures the spread of $f_X(x)$ about the mean.

- The square root of the variance σ_x, is called the standard deviation.
- The variance is given by:

$$\sigma_x{}^2 = u_2 = E\left[\left(X - E(X)\right)^2\right] = \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x) dx \tag{23}$$

We have:

$$\sigma_x{}^2 = E[X^2] - E[X]^2 \tag{24}$$

- This means that the variance can be determined by the knowledge of the first and second moments.

Example 3.4: $f_X(x) = \begin{cases} \frac{1}{b} e^{-\frac{(x-a)}{b}} & x \geq a \\ 0 & x < a \end{cases}$

Find $\sigma_x{}^2$?

Solu: $\sigma_x{}^2 = \int_a^{\infty} (x - \bar{x})^2 \frac{1}{b} e^{\frac{-(x-a)}{b}} \quad dx$

Let $\beta = x - \bar{x}, d\beta = dx, x = a \Rightarrow \beta = a - \bar{x}$

Then, $\sigma_x{}^2 = \int_{a-\bar{x}}^{\infty} \beta^2 \frac{1}{b} e^{\frac{-(x-\bar{x}+\bar{x}-a)}{b}} \quad d\beta$

$= \frac{e^{\frac{-(\bar{x}-a)}{b}}}{b} \int_{a-\bar{x}}^{\infty} \beta^2 e^{\frac{-\beta}{b}} \quad d\beta$

From table: $\int x^2 e^{cx} dx = e^{cx}[\frac{x^2}{c} - \frac{2x}{c^2} + \frac{2}{c^3}]$

$$\Rightarrow \sigma_x{}^2 = \frac{e^{\frac{-(\bar{x}-a)}{b}}}{b} \left[e^{\frac{-\beta}{b}} \quad (-b\beta^2 - 2b^2\beta - 2b^3) \right]_{a-\bar{x}}^{\infty}$$

$$= \frac{e^{\frac{-(\bar{x}-a)}{b}}}{b} [0 - e^{\frac{-(\bar{x}-a)}{b}} (-b(a-\bar{x})^2 - 2b^2(a-\bar{x}) - 2b^3)]$$

$$= (a - \bar{x})^2 - 2b(a - \bar{x}) - 2b^2$$

$since\ \bar{X} = E[X] = a + b\ (see\ example\ 3.1)$

$\sigma_x{}^2 = (a - a - b)^2 - 2b(a - a - b) - 2b^2 = b^2 + 2b^2 - 2b^2 = b^2$

Another solution: Use $\sigma_x{}^2 = E[X^2] - \bar{X}^2$

8.2.2 Skew

- The skew or third central moment is a measure of asymmetry of the density function about the mean.

$$u_3 = E\left[\left(X - E(X)\right)^3\right] = \int_{-\infty}^{\infty} (x - E[X])^3 f_X(x)dx \qquad (25)$$

$\Rightarrow$ $u_3 = 0$ If the density is symetric about the mean

Example 3.5. Compute the skew of a density function uniformly distributed in the interval [-1, 1].

Solution: $f_X(x) = \begin{cases} \frac{1}{2} & for\ -1 \leq x \leq 1 \\ 0 & otherwise \end{cases}$

$$E[X] = \int_{-\infty}^{+\infty} x f_X(x)dx = \int_{-1}^{+1} x.\frac{1}{2}dx = \frac{1}{2}\frac{x^2}{2}\Big|_{-1}^{1} = 0$$

$$u_3 = E\left[\left(X - E(X)\right)^3\right] = \int_{-\infty}^{\infty} (x - E[X])^3 f_X(x)dx = \int_{-1}^{1} (x)^3 \frac{1}{2}dx = \frac{1}{2}\frac{x^4}{4}\Big|_{-1}^{1} = 0$$

9. Functions that give moments

- The moments of a random variable *X* can be determined using two different functions: **Characteristic function** and the **moment generating function.**

9.1 Characteristic function

- The characteristic function of a random variable *X* is defined by:

$$\emptyset_X(\omega) = E[e^{j\omega x}] \tag{26}$$

- $j = \sqrt{-1}$ and $-\infty < \omega < +\infty$
- $\emptyset_X(\omega)$ can be seen as the Fourier transform (with the sign of ω reversed) of $f_X(x)$:

$$\emptyset_x(\omega) = \int_{-\infty}^{\infty} f_X(x) e^{jwx} dx \tag{27}$$

If $\emptyset_X(\omega)$ is known then density function $f_X(x)$ and the moments of X can be computed.

- The density function is given by:

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \emptyset_x(\omega) e^{-j\omega x} d\omega \tag{28}$$

- The moments are determined as follows:

$$m_n = (-j)^n \left.\frac{d^n \emptyset_X(\omega)}{d\omega^n}\right|_{\omega=0} \tag{29}$$

Differentiate *n* times with respect to ω and set $\omega = 0$ in the derivative

- Note that $|\emptyset_X(\omega)| \leq \emptyset_X(0) = 1$

Example 3.6: Let $f_X(x) = \begin{cases} \frac{1}{b} e^{\frac{-(x-a)}{b}} dx & x \geq a \\ 0 & x < a \end{cases}$

Evaluate the characteristic function and first moment.

Solu: $\emptyset_x(w) = \int_a^{\infty} \frac{1}{b} e^{\frac{-(x-a)}{b}} e^{jwx} dx$

$$= \frac{e^{a/b}}{b} \int_a^{\infty} e^{\left(-\frac{1}{b}+jw\right)x} dx = \frac{e^{a/b}}{b(-\frac{1}{b}+jw)} e^{-\left(\frac{1}{b}+jw\right)x} |_a^{\infty}$$

$$= \frac{e^{a/b}(0-e^{\left(-a/b+jaw\right)})}{-1+jbw} = \frac{e^{jaw}}{1-jbw}$$

$$m_1 = -j \frac{d\emptyset_x(w)}{dw} |_{w=0}$$

$$= -j \frac{[ja\, e^{jaw}(1-jbw) - e^{jaw}(-jb)]}{(1-jbw)^2} |_{w=0}$$

$$= -j \frac{[ja + jb]}{1} = -j^2[a+b] = a+b$$

9.2 Moment generating function

- The moment generating function is given by:

$$M_X(v) = E[e^{vx}] = \int_{-\infty}^{\infty} f_X(x) e^{vx} dx \tag{30}$$

Where v is a real number: $-\infty < v < \infty$

- Then the moments are obtained from the moment generating function using the following expression:

$$m_n = \left. \frac{d^n M_X(v)}{dv^n} \right|_{v=0} \tag{31}$$

⇨ Compared to the characteristic function, the moment generating function may not exist for all random variables.

Example 3.7: Compute $M_X(v)$ and m_1 for the exponential r.v.

Solu: $M_X(v) = \int_a^{\infty} \frac{1}{b} e^{\frac{-(x-a)}{b}} e^{vx} dx$

$$= \frac{e^{av}}{1-bv}$$

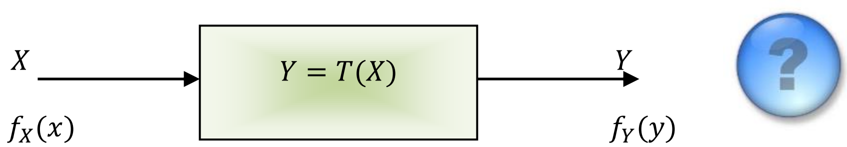
$$m_1 = \frac{ae^{av}(1-bv) + e^{av}b}{(1-bv)}|_{v=0} = \mathrm{a} + \mathrm{b}$$

10 Transformation of a random variable

- A random variable X can be transformed into another r.v. Y by:

$$Y = T(X) \qquad (32)$$

- Given $f_X(x)$ and $F_X(x)$, we want to find $f_Y(y)$, and $F_Y(y)$,
- We assume that the transformation T is continuous and differentiable.



10.1 Monotonic transformation

- A transformation T is said to be monotonically increasing $T(x_1) < T(x_2)$ for any $x_1 < x_2$.
- T is said monotonically decreasing if $T(x_1) > T(x_2)$ for any $x_1 < x_2$.

10.1.1 Monotonic increasing transformation

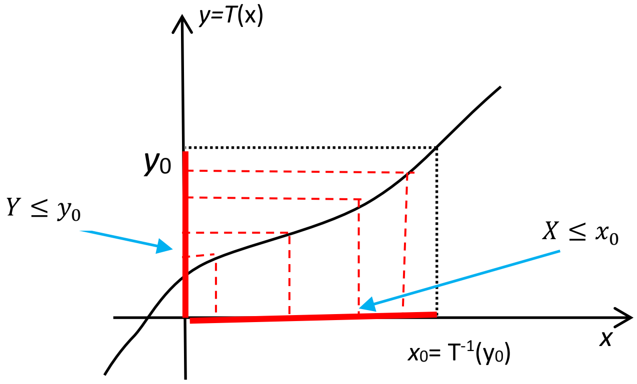


Figure 5. Monotonic increasing transformation

- In this case, for particular values x_0 and y_0 shown in figure 1, we have:

$$y_0 = T(x_0) \tag{33}$$

and

$$x_0 = T^{-1}(y_0) \tag{34}$$

- Due to the one-to-one correspondence between *X* and *Y*, we can write:

$$F_Y(y_0) = P\{Y \leq y_0\} = P\{X \leq T^{-1}(y_0)\} = F_X(x_0) \tag{35}$$

$$F_Y(y_0) = \int_{-\infty}^{y_0} f_Y(y)dy = \int_{-\infty}^{x_0} f_X(x)dx \tag{36}$$

- Differentiating both sides with respect to y_0 and using the expression $x_0 = T^{-1}(y_0)$, we obtain:

$$f_Y(y_0) = f_x[T^{-1}(y_0)]\frac{d\,T^{-1}(y_0)}{dy_0} \tag{37}$$

- This result could be applied to any y_0, then we have:

$$f_Y(y) = f_X[T^{-1}(y)]\frac{d\,T^{-1}(y)}{dy} \tag{38}$$

- Or in compact form:

$$f_Y(y) = f_x(x)\frac{dx}{dy}\bigg|_{x=T^{-1}(y)} \tag{39}$$

10.1.2 Monotonic decreasing transformation

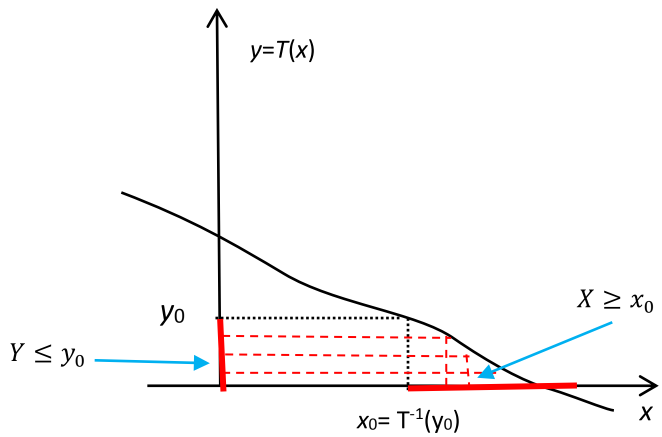


Figure 6. Monotonic decreasing transformation

- From Figure 2, we have

$$F_Y(y_0) = P\{Y \le y_0\} = P\{X \ge x_0\} = 1 - F_X(x_0) \tag{40}$$

$$F_Y(y_0) = \int_{-\infty}^{y_0} f_Y(y)dy = 1 - \int_{-\infty}^{x_0} f_X(x)dx \tag{41}$$

- Again Differentiating with respect to y_0, we obtain:

$$f_Y(y_0) = -f_Y[T^{-1}(y_0)]\frac{dT^{-1}(y_0)}{dy_0} \tag{42}$$

- As the slope of $T^{-1}(y_0)$ is negative, we conclude that for both types of monotonic transformation, we have:

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| \quad and \quad x = T^{-1}(y) \tag{43}$$

Example 3.8: Let Y=aX+b. Find $f_Y(y)$ given that $f_X(x)$is Gaussian r.v. with mean a_x and standard deviation σ_x.

Solu: $Y = aX + b \Rightarrow X = \frac{Y-b}{a}$ and $\frac{dx}{dy} = \frac{1}{a}$

$$\Rightarrow f_Y(y) = f_X\left(\frac{Y-b}{a}\right)\left|\frac{1}{a}\right|$$

When $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{\frac{-(x-ax)^2}{2\sigma_x{}^2}}$

Then $f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{\frac{-\left|\frac{y-b}{a}-a_x\right|^2}{2\sigma_x{}^2}}\left|\frac{1}{a}\right|$

$$= \frac{1}{|a|\sqrt{2\pi}\sigma_x} e^{\frac{-[y-(b+aa_x)]^2}{2a^2\sigma_x{}^2}}$$

Y is also Gaussian with mean and variance:

$$a_Y = aa_x + b \quad and \quad \sigma_Y{}^2 = a^2\sigma_x{}^2$$

a. Non-monotonic transformation

- In general, a transformation could be non monotonic as shown in figure 3

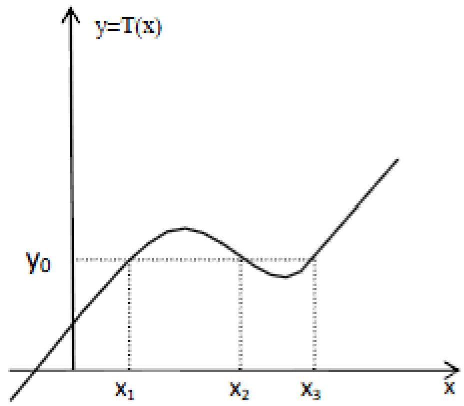


Figure 7. A non-monotonic transformation

- In this case, more than one interval of values of X that correspond to the event $P(Y \leq y_0)$
- For example, the event represented in figure 7 corresponds to the event $\{X \leq x_1 \; and \; x_2 \leq X \leq x_3\}$.

- In general for **non-monotonic transformation**:

$$f_Y(y) = \sum_{j=1}^{N} \frac{f_X(x_j)}{\left|\frac{dT(x)}{dx}\right|_{x=x_j}} \tag{44}$$

Where x_j , j= 1,2,. . .,N are the real solutions of the equation $T(x) = y$

Example 3.9: Let Y=T(x)=cX² ; c > 0 .

Given $f_X(x), find\ f_Y(y)$?

Solu: $Y = cX^2 \Rightarrow x_1 = \sqrt{\frac{y}{c}}, \quad x_2 = -\sqrt{\frac{y}{c}}$

$y' = 2cx$

$$f_Y(y) = \frac{f_X\left(\sqrt{\frac{y}{c}}\right)}{\left|2c\sqrt{\frac{y}{c}}\right|} + \frac{f_X\left(-\sqrt{\frac{y}{c}}\right)}{\left|2c\sqrt{\frac{y}{c}}\right|}$$

$$f_Y(y) = \frac{f_X\left(\sqrt{\frac{y}{c}}\right)+f_X\left(-\sqrt{\frac{y}{c}}\right)}{2\sqrt{cy}}; y \geq 0$$