Chapter 1: Probability Theory

Objectives of the chapter

1. **Introduce basic concept of sets**
2. **Introduce basic concepts of probability**
3. **Introduce some useful counting methods**

CLO1	Explain basic concepts probability, joint probability, conditional probability, independence, total probability, and Bayes' rule.

1. Set definitions

- A set can be defined as a collection of objects. Sets are generally denoted by capital letters as: A, B, C, ...

- The individual objects forming the set are called "**elements**" or "**members**". They are generally denoted by lower case letters as: a,b, c,...

- If an element g belongs to a set G, we write:

$$g \in G \tag{1}$$

  Otherwise, we say g is not a member of G, we write:

$$g \notin G \tag{2}$$

- A set is specified by the content of two braces: $\{\cdot\}$.

- **Representation of sets:**

  - Tabular method: the elements are enumerated explicitly. For example: A={3,4,5,6}.
  - Rule method: the content of the set is specified using a rule. This representation is more convenient when the set is large. For example:

$$G = \{g \mid gis\ an\ integerand\ 3 \leq g \leq 6\} \tag{3}$$

Such that

- **Countable and uncountable sets:** A set is called to be **"countable"** if its elements can be put in one-to-one correspondence with the integers 1,2,..etc.Otherwise, it is called "**uncountable**".

- **Empty set:** A set $G$is said to be empty, if it has no elements. It is also called null set and it is denoted by $\emptyset$.

- **Finite and infinite sets:** A finite set is either empty set or has elements that can be counted, with the counting process terminating. If a set is not finite it is called infinite.

- **Subset:** Given two sets A and B, if every element of A is also an element of B, A is said to be contained in $\boldsymbol{B}$. A is known as a subset of B. We write:

$$A \subseteq B \tag{4}$$

is contained

- **Proper subset:** If at least one element in $B$is not in A, then$A$ is a proper subset of B, denoted by

$$A \subset B \tag{5}$$

- **Disjoint sets:** If two sets A and B have no common elements, then they are called disjoint or mutually exclusive.

Example 1:

Let us consider the following four sets:

$A = \{ 1,3,5,7\}$ $\qquad$ $D = \{ 0 \}$

$B = \{ 1,2,3, \dots .\}$ $\qquad$ $E = \{ 2,4,6,8,10,12,14\}$

$C = \{c | cisrealand\ 0.5 < c \leq 8.5\}$ $\qquad$ $F = \{f | f\ is\ real\ and\ -5 < f \leq 12\}$

Illustrate the previous concepts using the sets A, B, C, D, E, F.

Set *A*	Relation between set *A* and set *B*	Set *B*
Tabular or rule countable or uncountable Finite or infinite Empty	(subset, proper subset, mutually exclusive) ⟺	Tabular or rule Finite or infinite countable or uncountable Empty

Solution:

- **The set A is tabularly specified, countable, and finite.**
- **Set A is contained in sets B, C and F.**
- **The set B is tabularly specified and countable, but is infinite.**
- **Set C is rule-specified, uncountable, and infinite.**
- **Sets D and E are countably finite.**
- **Set F is uncountably infinite.**
- $\boldsymbol{C \subset F,\ D \subset F}$**, E** $\boldsymbol{\subset B.}$
- **Sets $\boldsymbol{B}$ and $\boldsymbol{F}$ are not sub sets of any of the other sets or of each other.**
- **Sets $\boldsymbol{A}$, $\boldsymbol{D}$ and $\boldsymbol{E}$ are mutually exclusive of each other.**

- **Universal set:** The set of all elements under consideration is called the universal set, denoted S. All sets (of the situation considered) are subsets of S.

  If we have a set S with n elements, then there are 2^n subsets.

  In case of rolling die, the universal set is $S = \{1,2,3,4,5,6\}$ and the number of subsets is 2^6=64 subsets.

Example 2:

Determine the subsets of the following universal set $S = \{\ 1,2,3,4\}$

Solution:

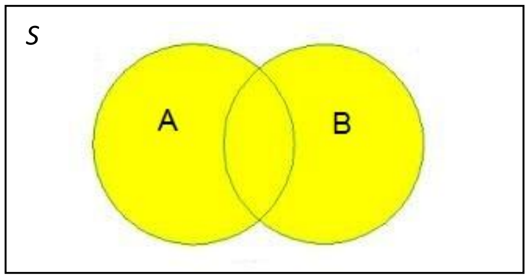
The universal set is $\boldsymbol{S} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\}$ and the number of subsets is $\mathbf{2^4}$=16 subsets.

1	$\emptyset$	9	$\{2,3\}$
2	$\{1\}$	10	$\{2,4\}$
3	$\{2\}$	11	$\{3,4\}$
4	$\{3\}$	12	$\{1,2,3\}$
5	$\{4\}$	13	$\{1,\ 3,4\}$
6	$\{1,2\}$	14	$\{1,2,4\}$
7	$\{1,3\}$	15	$\{2,3,4\}$
8	$\{1,4\}$	16	$\{1,2,3,4\}$

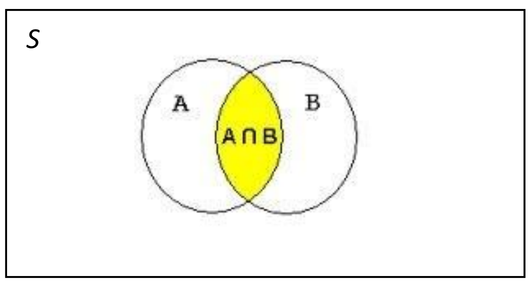
2. Set Operations

- **Venn diagram:** is a graphical representation of sets to help visualize sets and their operations.

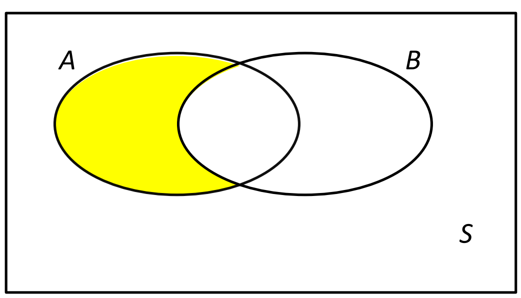
- **Union:** set of all elements that are members of A or B or both and is denoted by $A \cup B$.



- **Intersection:** set of all elements which belong to both $A$and B and is denoted by $A \cap B$

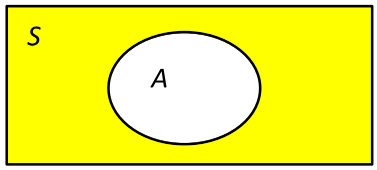


- **Difference:** Set consisting of all elements in A which are not in B and is denoted as $A - B$



- **Complement:** The set composed of all members in $S$and not in A is the complement of A and denoted $\bar{A}$. Thus

$$\bar{A} = S - A \tag{6}$$



It is easy to see that $\overline{\emptyset} = S,\ \overline{S} = \emptyset,\ \ A \cup \overline{A} = S,\ \ and\ A \cap \overline{A} = \emptyset$

Example 3:

Let us illustrate these concepts on the following four sets

$$S = \{\, a \mid a \text{ is an integer and } 1 \le a \le 12\}$$

$A = \{1,3,5,12\}$

$B = \{2,6,7,8,9,10,11\}$

$C = \{1,3,4,6,7,8\}$

Solution:

- **Unions and intersections**

$\boldsymbol{A \cup B = \{1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12\}}$ $\boldsymbol{A \cap B = \emptyset}$

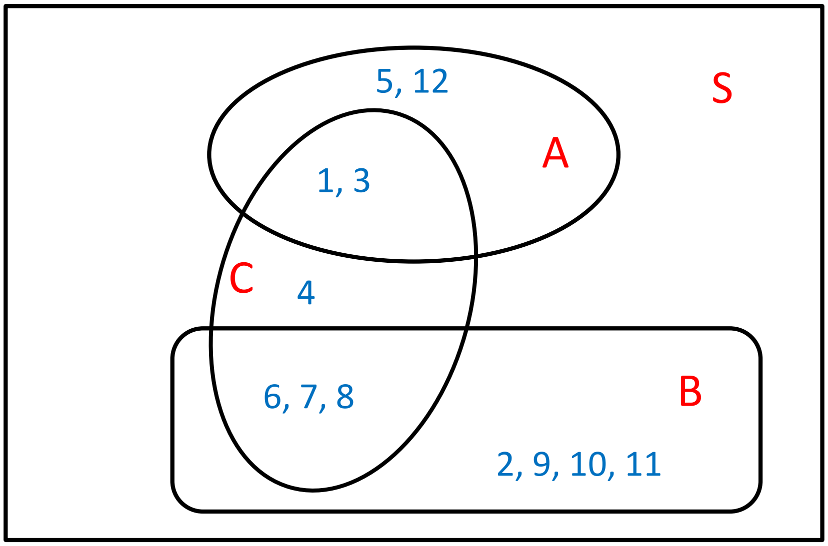
$\boldsymbol{A \cup C = \{1, 3, 4, 5, 6, 7, 8, 12\}}$ $\boldsymbol{A \cap C = \{1, 3\}}$

$\boldsymbol{B \cup C = \{1, 2, 3, 4, 6, 7, 8, 9, 10, 11\}}$ $\boldsymbol{B \cap C = \{6, 7, 8\}}$

- **Complements** $\boldsymbol{\overline{A} = \{2, 4, 6, 7, 8, 9, 10, 11\}}$

$\boldsymbol{\overline{B} = \{1, 3, 4, 5, 12\}}$

$\boldsymbol{\overline{C} = \{2, 5, 9, 10, 11, 12\}}$



- **Algebra of sets:**
  - **Commutative law:** $A \cap B = B \cap A$

    $$A \cup B = B \cup A$$

  - **Distributive law:** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

    $$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

  - **Associative law:** $(A \cup B) \cup C = A \cup (B \cup C) = A \cup B \cup C$

    $$(A \cap B) \cap C = A \cap (B \cap C) = A \cap B \cap C$$

  - **De Morgan's Law:** $\overline{A \cup B} = \overline{A} \cap \overline{B}$

    $$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

3. Probability

- We use probability theory to develop a mathematical model of an experiment and to predict the outcome of an experiment of interest.
- A single performance of the experiment is called a **trial** for which there is an **outcome**.
- In building the relation between the set theory and the notion of probability, we call the set of all possible distinct outcomes of interest in a particular experiment as the **sample space** S
- The sample space S may be different for different experiments.
- The sample space S can be discrete or continuous, countable or uncountable, finite or infinite.
- An **event** is a particular outcome or a combination of outcomes.
- An event is a subset of the sample space S.

Probability definition and axioms

- Let A an event defined on the sample space S. The probability of the event A denoted as $P(A)$ is a function that assigns to A a real number such that:

✓ **Axiom1:** $P(A) \geq 0$ (7)

✓ **Axiom2:** $P(S) = 1$ (8)

Certain event

✓ **Axiom3:** if we have N events $A_n, n = 1, 2, \dots, N$ defined on the sample space S, and having the propriety: $A_m \cap A_n = \emptyset$ for $m \neq n$ (mutually exclusive events). Then:

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n) \quad (9)$$

Or $$P(\cup_{n=1}^{N} A_n) = \sum_{n=1}^{N} P(A_n) \quad (10)$$

Some Properties:

- For every event A, its probability is between 0 and 1:

$$0 \leq P(A) \leq 1 \quad (11)$$

- The probability of the impossible event is zero

$$P(\emptyset) = 0 \tag{12}$$

- If $\bar{A}$ is the complement of A, then:

$$P(\bar{A}) = 1 - P(A) \tag{13}$$

- **To model a real experiment mathematically, we shall :**
  - Define the sample space.
  - Define the events of interest.
  - Assign probabilities to the events that satisfy the probability axioms.

Example 4:

An experiment consists of observing the sum of two six sided dice when thrown randomly. Develop a model for the experiment.

- Determine sample space S
- Let the event A be: "the sum events is 7"
- Let the event B be: "8< $sum \leq 11$".

  Determine $P(A), P(B),\ P(\bar{A}), P(\bar{B})$.

Solution

The sample space: if one experiments can result in any of m possible outcomes and if another experiment can result in any of n possible outcomes, then there are nm possible outcomes of the two experiments (basic principle of counting). The sample space consists of $\mathbf{6^2 = 36}$ different outcomes.

$$S = \left\{ \begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \color{red}{(1,6)} \\ (2,1) & (2,2) & (2,3) & (2,4) & \color{red}{(2,5)} & (2,6) \\ (3,1) & (3,2) & (3,3) & \color{red}{(3,4)} & (3,5) & \color{green}{(3,6)} \\ (4,1) & (4,2) & \color{red}{(4,3)} & (4,4) & \color{green}{(4,5)} & \color{green}{(4,6)} \\ (5,1) & \color{red}{(5,2)} & (5,3) & \color{green}{(5,4)} & \color{green}{(5,5)} & \color{green}{(5,6)} \\ \color{red}{(6,1)} & (6,2) & \color{green}{(6,3)} & \color{green}{(6,4)} & \color{green}{(6,5)} & (6,6) \end{matrix} \right\}$$

Let events A= {sum = 7}, B= {8$< sum \leq 11$}.

In probability assignment, if the dice are not biased, then P(each outcome)=1/36.

To obtain p(A) and P(B), note that the outcomes are mutually exclusive: therefore, axiom 3 applies:

P(A) = P($\bigcup_{i=1}^{6}$ $S_{i,7-i}$) =6($\frac{1}{36}$) = $\frac{1}{6}$

P(B) = 9($\frac{1}{36}$)= $\frac{1}{4}$

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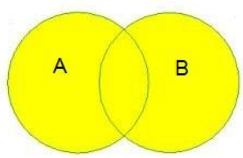
4. Joint and conditional probability

Joint probability

- When two events *A* and *B* have some elements in common (not mutually exclusive), then axiom3 cannot be applied.
- The probability P($A \cap B$) is called the joint probability for the events *A* and *B which intersect in sample space.*

$$P(A \cap B)=P(A)+P(B)-P(A \cup B) \tag{14}$$

Equivalently:



$$P(A \cup B)=P(A)+P(B)-P(A \cap B)$$

Conditional probability

- Given some event *B* with nonzero probability *P*(*B*)>0
- We defined, the conditional probability of an event A given *B*, by:

$$P(A|B)=\frac{P(A \cap B)}{P(B)} \tag{15}$$

- $P(A|B)$ is the probability that *A* will occur given that *B* has occurred.

- If the occurrence of event B has no effect on A, we say that A and B are independent events. In this case,

$$\mathrm{P}(A|B) = P(A) \tag{16}$$

Which means that:

$$P(A \cap B) = P(A)P(B) \tag{17}$$

Example 4

In a box there are 100 resistors having resistance and tolerance as shown below:

	Tolerance		
Resistance(Ω)	5%	10%	Total
22	10	14	24
47	28	16	44
100	24	8	32
Total	62	38	100

Let a resistor be selected from the box and define the events:

A = 'Draw 47 Ω resistor'

B = 'Draw resistor with 5% tolerance'

C = 'Draw 100 Ω resistor'

Find P(A), P(B), P(C), $P(A \cap B)$, $P(A \cap C)$, $P(B \cap C)$, $P(A|B)$, $P(A|C)$, $P(B|C)$.

Solution

P(A)=P(47 Ω)=44/100=0.44. P(B)=P(5%)=62/100=0.62 P(C)=P(100 Ω)=32/100=0.32

Joint probabilities are:

$$P(A \cap B) = P(47\Omega \cap 5\%) = \frac{28}{100} = 0.28$$

$$P(A \cap C) = P(47\Omega \cap 100\Omega) = 0$$

$$P(B \cap C) = P(5\% \cap 100\Omega) = \frac{24}{100} = 0.24$$

The conditional probabilities become:

$P(A\ /\ B)$ =P(47Ω / 5%) is the probability of drawing a 47 Ω resistor given that the resistor drawn is 5%.

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{28}{62}$$

$$P(A/C) = \frac{P(A \cap C)}{P(C)} = 0$$

$$P(B/C) = \frac{P(B \cap C)}{P(C)} = \frac{24}{32}$$

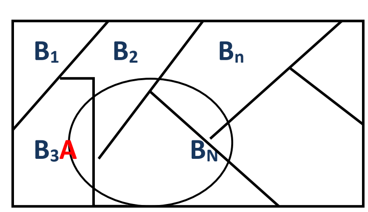
Total probability

- Suppose we are given n mutually exclusive events B_n, n= 1…….,N such that:

$$\cup_{n=1}^{N} Bn = S \tag{18}$$

and

$B_m \cap B_n$ = Ø for m≠ n



- The **total probability of** an event A defined on the sample space S can be expressed in terms of conditional probabilities as follows:

$$P(A) = \sum_{n=1}^{N} P(A|B_n)P(B_n) \tag{19}$$

<u>Prove</u>: since $A = A \cap S = A \cap (\bigcup_{n=1}^{N} Bn) = \bigcup_{n=1}^{N}(A \cap Bn)$

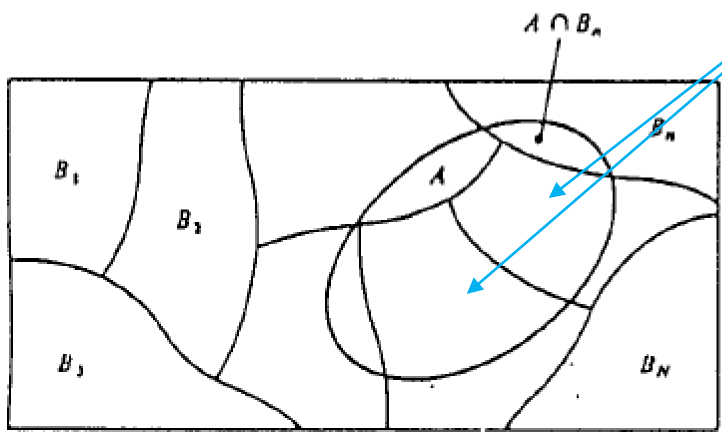
As shown in the diagram, $A \cap Bn$ events are mutually exclusive; therefore:

$\mathrm{P}(\mathrm{A}) = \mathrm{P}[\bigcup_{n=1}^{N}(A \cap Bn)] = \sum_{n=1}^{N} P(A \cap Bn)$

Since $\mathrm{P}(A \cap Bn) = P(A|Bn)\, P(Bn)$ #

$A \cap B_m$ *and* $A \cap B_n$

Are mutually exclusive



Bayes' Theorem:

- The Bayes rule expresses a conditional probability in terms of other conditional probabilities, we have:

$$P(B_n|A) = \frac{P(B_n \cap A)}{P(A)} \tag{20}$$

$$P(A|B_n) = \frac{P(A \cap B_n)}{P(Bn)} \tag{21}$$

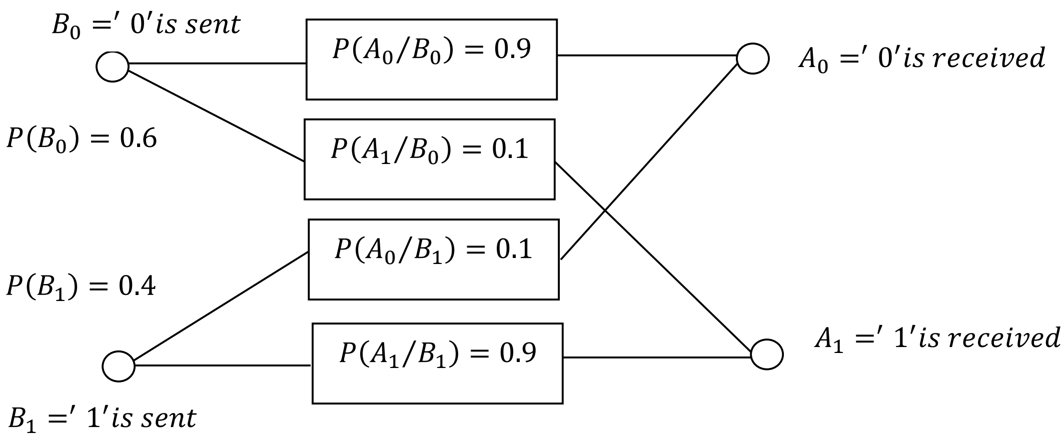
Therefore one form of the Bayes theorem is given by equating these two expressions:

$$P(Bn|A) = \frac{P(A|B_n)P(B_n)}{P(A)} \qquad (22)$$

which can be written also as another form:

$$P(Bn|A) = \frac{P(A|B_n)P(B_n)}{P(A|B_1)P(B_1)+,\ldots\ldots P(A|B_N)P(B_N)} \qquad (23)$$

Example 5: A binary Communication system is described as:



Find:

a) $P(A_0)$ ('0' is received).

b) $P(A_1)$ ('1' is received).

c) $P(B_0/A_0), P(B_0/A_1), P(B_1/A_0), P(B_1/A_1)$.

Solution:

Solu:

Total probability

a) $P(A_0)=P(A_0| B_0)\, P(B_0) + P(A_0| B_1)\, P(B_1)$

$= 0.9(0.6) + 0.1(0.4) = 0.58$

b) $P(A_1) = P(A_1| B_0)\, P(B_0) + P(A_1| B_1)\, P(B_1)$

$= 0.1(0.6) + 0.9(0.4) = 0.42$

Note that A_0 and A_1 are mutually exclusive and $P(A_0) + P(A_1) = 1$

c) $P(B_0| A_0) = \frac{P(A_0| B_0)\, P(B_0)}{P(A_0)} = \frac{0.9\ (0.6)}{0.58} = 0.931$

$P(B_0| A_1) = \frac{P(A_1| B_0)\, P(B_0)}{P(A_1)} = \frac{0.1(0.6)}{0.42} = 0.143$

$P(B_1| A_0) = \frac{P(A_0| B_1)\, P(B_1)}{P(A_0)} = \frac{0.1\ (0.4)}{0.58} = 0.069$

$P(B_1| A_1) = \frac{P(A_1| B_1)\, P(B_1)}{P(A_1)} = \frac{0.9\ (0.4)}{0.42} = 0.857$

Bayes' Theorem

Note that $P(B_0| A_1)$ and $P(B_1| A_0)$ are probabilities of error and $P(B_0| A_0)$ and $P(B_1| A_1)$ are probabilities of correct transmission.

5. Independent Events

- Two events *A* and *B* are said to be independent if the occurrence of one event is not affected by the occurrence of the other. That is:

$$P(A|B) = P(A) \tag{24}$$

And we also have

$$P(B|A) = P(B) \tag{25}$$

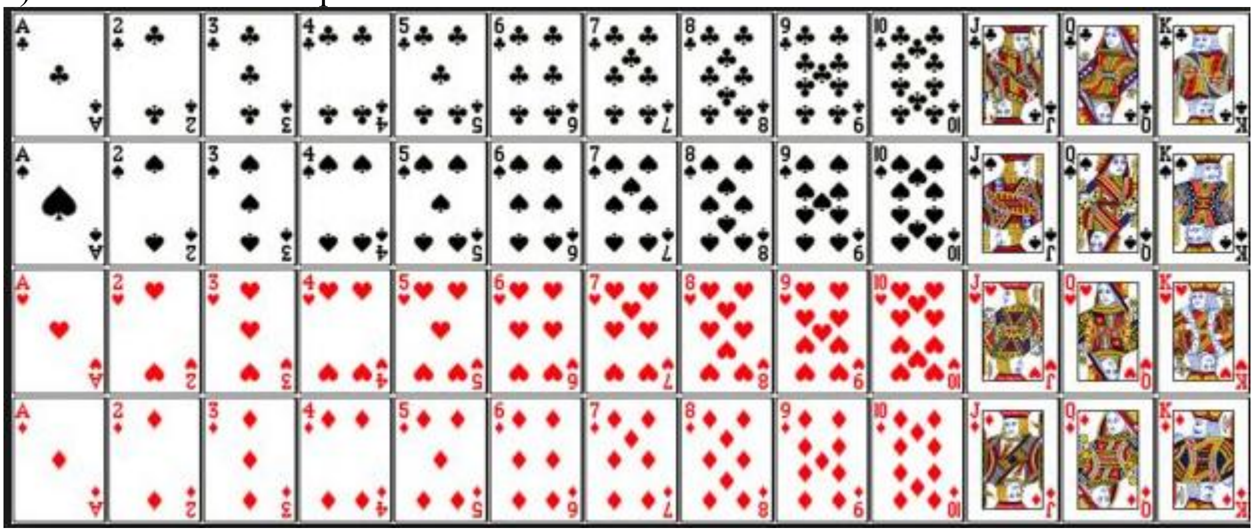
Since

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \Longrightarrow P(A \cap B) = P(A)P(B) \qquad \text{(joint occurrence, intersection)} \tag{26}$$

- Note that for mutually exclusive events **P (A ∩ B) = 0**
Therefore, for $P(A) \neq 0,\ P(B) \neq 0$, A and B cannot be both mutually exclusive ($A \cap B = \emptyset$), and independent ($A \cap B \neq \emptyset$).

<u>Example 1.5:</u>
One card is selected from 52 card deck. Define events A= “select a king”, B=select Jack or queen” and C= “select a heart”

Find:
a) P(A), P(B), P(C)
b) $P(A \cap B), P(B \cap C), P(A \cap C)$.
c) Are the events independent?



Solu:

a) $\mathrm{P(A)} = \frac{4}{52}$, $\mathrm{P(B)} = \frac{8}{52}$, $\mathrm{P(C)} = \frac{13}{52}$

It is not possible to simultaneously select a king and a jack or a queen.

b) $\mathrm{P}(A \cap B) = 0$, $\mathrm{P}(A \cap C) = \frac{1}{52}$, $\mathrm{P}(B \cap C) = \frac{2}{52}$

We determine whether A, B, and C are independent by pairs.

c) $\mathrm{P}(A \cap B) = 0 \neq \mathrm{P(A)\,P(B)} \Longrightarrow$ A and B are not independent

$\mathrm{P}(A \cap C) = \frac{1}{52} = \mathrm{P(A)\,P(C)} \Longrightarrow$ A and C are independent

$\mathrm{P}(B \cap C) = \frac{2}{52} = \mathrm{P(B)\,P(C)} \Longrightarrow$ B and C are independent

- In case of multiple events, they are said to be independent if all pairs are independent and:

$$\mathrm{P}(A1 \cap A2 \cap A3) = P(A1)\,P(A2)\,P(A3)$$

6. Combined Experiments

- A combined experiment consists of forming a single experiment by suitably combining individual experiments.
- These experiments are called *sub-experiments*

- If we have N sample spaces S_n ; n=1,2,.....N having elements s_n then the combined sample space is defined as:

$$S=S_1\times S_2\times\ldots\times S_N \tag{27}$$

Example 6:

Let us consider the two following sub-experiments:

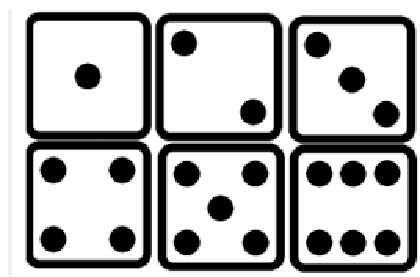
- Flipping a coin
- Rolling of single die

Determine the sample space S_1 and S_2 corresponding to these two sub-experiments.

Determine the combined sample space S.

Solution:





Solu: for flipping a coin: $S_1=\{H,T\}$

For rolling a die: $S_2=\{1,2,3,4,5,6\}$

$S=S_1XS_2=\{(H,1), (H,2), (H,3), (H,4), (H,5), (H,6), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)\}$

Example 7:

We flip a coin twice. What is the combined sample space S

Solution:

$\boldsymbol{S_1=\{H,T\}}$

$\boldsymbol{S_2=\{H,T\}}$

$\boldsymbol{S = \{(H,H),(H,T),(T,H),(T,T)\}}$

7. Some counting methods

Each digit has 3 possibilities

- Count the number of words of length k with n letters.

  <u>Ex:</u> n=3, {A, B, C}. and k=5, <u>A</u> <u>B</u> <u>B</u> <u>A</u> <u>C</u> the number #= 3^5 more generally #=n^k .

  3 3 3 3 3

- Count the number of words of length k from alphabet of k letters with no allowed repetition (i.e. Permutation of k objects).

  <u>**Ex**</u>: n=5 {A, B, C, D, E}, k=5, <u>D</u> <u>B</u> <u>E</u> _ _ # = k (k-1) (k-2) … (2) (1) = k!

  5 4 3 2 1

- Number of words of length k from alphabet with n letters, repetition not allowed (**permutation** ordering is important here)

$$\# = P_k^n = \mathrm{n}\,(\mathrm{n}-1)\,(\mathrm{n}-2)\ \dots\ (\mathrm{n}-\mathrm{k}+1) = \frac{n!}{(n-k)!}$$

1 2 3 … k

- If order of elements in a sequence is not important (take one element from all the permuted elements), then the number of possible sequences is called **combinations**, which equals P_k^n divided by the number of permutations (orderings) of k elements $P_k^k = k!$. The number of combinations of k elements taken from n elements $\binom{n}{k}$ is:

$$C_k^n = \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

$\binom{n}{k}$ is called the binomial coefficients.

Example 8:

How many permutations for four cards taken from 52 cards?

<u>Solu</u>: $P_4^{52} = \frac{52!}{(52-4)!} = 52(51)(50)(49) = 6{,}497{,}400$

Example 9:

A team of 3 players is to be selected from 5 players, how many teams can be chosen?

Solu: $\binom{5}{3} = \frac{5!}{3!\,2!} = 10$

Example 10:

A number is composed of 5 digits. How many way are there for this number?

Solu: no. of ways = 10 X 10 X 10 X 10 X10 = 10^5

8. Bernoulli Trials

- These type of experiments are characterized by two possible outcomes: A and $\bar{A}$.

  For example: flipping a coin, hitting or missing a target, receiving 0 or 1.

- Let $P(A) = p$, Then $P(\bar{A})$= 1-p.

If the experiment is repeated N times, then the probability that event ***A*** occurs ***k*** times (regardless of ordering) equals the probability of this sequence multiplied by its number. In this case, $\bar{A}$ will occur ***N-k*** times and the probability of this sequence (one sequence) is:

$$\underbrace{P(A)\,P(A)\,\dots\,P(A)}_{k\ times}\ \underbrace{P(\bar{A})\,P(\bar{A})\dots\,P(\bar{A})}_{N-k\ times} = P^k(1-P)^{N-k}$$

- There are other sequences that will yield k events A and N-k events $\bar{A}$, From a combinatorial analysis, the number of sequences where A occurs k times in N trials is:

$$\binom{N}{k} = \frac{N!}{k!\,(N-k)!}$$

Finally we obtain the probability: P (A occurs k times) = $\binom{N}{K}p^k(1-p)^{N-k}$

Example 12:

A submarine will sink a ship if two or more rockets hit the ship. If the submarine fires 3 rockets and P(hit) = 0.4 for each rocket, what is the probability that the ship will be sunk?

Solution:

$P(no\ hits) = \binom{3}{0} 0.4^0(1-0.4)^3 = 0.216$

$P(1\ hit) = \binom{3}{1} 0.4^1(1-0.4)^2 = 0.432$

$P(2\ hits) = \binom{3}{2} 0.4^2(1-0.4)^1 = 0.288$

$P(3\ hits) = \binom{3}{3} 0.4^3(1-0.4)^0 = 0.064$

P (ship sunk) = P (two hits and more) = P (2 hits) + P (3 hits) = 0.352