Diagonalization of Matrix

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Eigenvalue and Eigenvector

Definition

If $A \in M_n(\mathbb{R})$ and $\lambda \in \mathbb{R}$.
λ is called an eigenvalue of the matrix A if there is $X \in \mathbb{R}^n \setminus \{0\}$ such that

$$AX = \lambda X.$$

The corresponding nonzero X are called eigenvectors of the matrix A.

Example

If A is the matrix $A = \begin{pmatrix} 1 & 1 \\ -3 & 5 \end{pmatrix}$, then the vector $X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector for A because $AX = 2X$. The corresponding eigenvalue is $\lambda = 2$.

Remark Note that if $AX = \lambda$ and c is any real number, then $A(cX) = cAX = c(\lambda X) = \lambda(cX)$. Then, if X is an eigenvector of A, then so is cX for any nonzero number c.

The eigenvalue equation can be rewritten as $(A - \lambda I)X = 0$. The eigenvalues of A are the values of λ for which the above equation has nontrivial solutions. There are nontrivial solutions if and only if $\det(A - \lambda I) = 0$.

Theorem

If $A \in M_n(\mathbb{R})$ and $\lambda \in \mathbb{R}$. λ is an eigenvalue the matrix A if and only if $\det(A - \lambda I) = 0$.

Definition

If $A \in M_n(\mathbb{R})$, the polynomial

$$q_A(\lambda) = |A - \lambda I|$$

is called the characteristic polynomial of the matrix A and the equation $q_A(\lambda) = 0$ is called the characteristic equation of A. The eigenvalues of A are the roots of its characteristic polynomial.

Example

Find all of the eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$.

Compute the characteristic polynomial $q_A(\lambda) = \begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} = (\lambda+1)(\lambda-4)$. The roots of $q_A(\lambda)$ are -1 and 4. $X_1 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ is an eigenvector for A with respect to the eigenvalue -1 and $X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector for A with respect to the eigenvalue 4.

Example

Find the eigenvalues of the following matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}, A = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}, A = \begin{pmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{pmatrix}.$$

Definition

If A is a matrix with characteristic polynomial p(),
If A is a matrix with characteristic polynomial $q_A(\lambda)$, the multiplicity of a root λ of q_A is called the algebraic multiplicity of the eigenvalue λ.

Example Let $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$. The characteristic function of the matrix A is

$$q_A(\lambda) = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ 1 & \lambda - 1 & 1 \\ -1 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 1)^2(2 - \lambda).$$

The eigenvalue $\lambda = 1$ has algebraic multiplicity 2, while $\lambda = 2$ has algebraic multiplicity 1.

Definition

Let $A \in M_n(\mathbb{R})$ and λ an eigenvalue of the matrix A. The set

$$E_\lambda = \{X \in \mathbb{R}^n;\ AX = \lambda X\}$$

is called the eigenspace of A associated to the eigenvalue λ.

Remark

If λ is an eigenvalue of the matrix $A \in M_n(\mathbb{R})$, then $E_\lambda = \{X \in \mathbb{R}^n;\ AX = \lambda X\}$ is vector sub-space of $\mathbb{R}^n$. Its dimension is called the the geometric multiplicity of λ.

The geometric multiplicity of λ is the number of linearly independent eigenvectors corresponding to λ.

Theorem

The geometric multiplicity of an eigenvalue is less than or equal to its algebraic multiplicity.

Definition

A matrix that has an eigenvalue whose geometric multiplicity is less than its algebraic multiplicity is called defective.

Theorem

If $A \in M_n(\mathbb{R})$ and $X_1, \ldots, X_m$ are eigenvectors for different eigenvalues $\lambda_1, \ldots, \lambda_m$, then $X_1, \ldots, X_m$ are linearly independent.

Proof

The proof is by induction. The result is true for $m = 1$.
Assume the result true for m and let $X_1, \ldots, X_{m+1}$ be eigenvectors for different eigenvalues $\lambda_1, \ldots, \lambda_{m+1}$.

If $a_1X_1 + \dots a_mX_m + a_{m+1}X_{m+1} = 0$, then $a_1\lambda_1X_1 + \dots a_m\lambda_mX_m + a_{m+1}\lambda_{m+1}X_{m+1} = 0$. Also we have $a_1\lambda_{m+1}X_1 + \dots a_m\lambda_{m+1}X_m + a_{m+1}\lambda_{m+1}X_{m+1} = 0$. Then $a_1(\lambda_1 - \lambda_{m+1})X_1 + \dots + a_m(\lambda_m - \lambda_{m+1})X_m = 0$. Since $(\lambda_j - \lambda_{m+1}) \neq 0$ for all $j = 1, \dots m$, then $a_1 = \dots = a_m = 0$ and so $a_{m+1} = 0$.

Definition

A matrix $A \in M_n(\mathbb{R})$ is called diagonalizable if there exists an invertible matrix $P \in M_n(\mathbb{R})$ such that the matrix $P^{-1}AP$ is diagonal .

Remark

If $X_1, \dots, X_n$ are the columns of the matrix P, then the columns of the matrix AP are: $AX_1, \dots, AX_n$.
Moreover if

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & \vdots \\ \vdots & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \dots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \lambda_n \end{pmatrix}$$

then the columns of the matrix PD are: $\lambda_1 X_1, \dots, \lambda_n X_n$.
Then $P^{-1}AP = D \iff PD = AP$ and the columns of the matrix P form a basis of $\mathbb{R}^n$ of eigenvectors of the matrix A.

Theorem

A matrix $A \in M_n(\mathbb{R})$ is diagonalizable if and only if it has n eigenvectors linearly independent. These vectors form a basis of the vector space $\mathbb{R}^n$.

Examples

Prove that the following matrices are diagonalizable and find an invertible matrix $P \in M_n(\mathbb{R})$ such that the matrix $P^{-1}AP$ is diagonal and find A^{15}.

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix},\ A = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix},\ A = \begin{pmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{pmatrix}.$$

Theorem

If $A \in M_n(\mathbb{R})$ and the characteristic function

$$q_A(\lambda) = C(\lambda - \lambda_1)^{m_1} \dots (\lambda - \lambda_p)^{m_p}$$

then A is diagonalizable if and only if the algebraic and geometric multiplicities are equal.

Remark

For example, if a matrix $A \in M_n(\mathbb{R})$ and has n different eigenvalues, then A is diagonalizable.

Example

Consider the matrix $A = \begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix}$. The characteristic polynomial of the matrix A is

$$q_A(\lambda) = \begin{vmatrix} 5-\lambda & 4 \\ -4 & -3-\lambda \end{vmatrix} = (1-\lambda)^2.$$

Then the matrix is not diagonalizable.

Example

Consider the matrix $A = \begin{pmatrix} -10 & -6 \\ 18 & 11 \end{pmatrix}$. The characteristic polynomial of the matrix A is

$$q_A(\lambda) = \begin{vmatrix} -10-\lambda & -6 \\ 18 & 11-\lambda \end{vmatrix} = (\lambda - 2)(1+\lambda).$$

Then the matrix is diagonalizable.
$E_{-1} = \langle(-2,3)\rangle$ and $E_2 = \langle(1,-2)\rangle$.
The diagonal matrix is $D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$
and the matrix P is $P = \begin{pmatrix} -2 & 1 \\ 3 & -2 \end{pmatrix}$.

Example

Consider the matrix $A = \begin{pmatrix} 5 & 0 & 4 \\ 2 & 1 & 5 \\ -4 & 0 & -3 \end{pmatrix}$. The characteristic polynomial of the matrix A is

$$q_A(\lambda) = \begin{vmatrix} 5-\lambda & 0 & 4 \\ 2 & 1-\lambda & 5 \\ -4 & 0 & -3-\lambda \end{vmatrix} = (1-\lambda)^3.$$

Then the matrix is not diagonalizable.

Example

Consider the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$. The characteristic polynomial of the matrix A is

$$q_A(\lambda) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ -1 & 1-\lambda & -1 \\ 1 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda).$$

$E_1 = \langle (0,1,0),(1,0,-1) \rangle$ and $E_2 = \langle (0,1,-1) \rangle$.
Then the matrix is diagonalizable.

the diagonal matrix is $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

and $P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}$.

Example

Consider the matrix $A = \begin{pmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

The characteristic polynomial of the matrix A is

$$q_A(\lambda) = \begin{vmatrix} 5-\lambda & -3 & 0 & 9 \\ 0 & 3-\lambda & 1 & -2 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix} = (5-\lambda)(3-\lambda)(2-\lambda)^2.$$

The matrix is diagonalizable if and only if the dimension of the vector space E_2 is 2.

$E_2 = \langle (1,1,-1,0), (-1,2,0,1) \rangle$.
Then the matrix A is diagonalizable.
$E_5 = \langle (1,0,0,0) \rangle$ and $E_3 = \langle (3,2,0,0) \rangle$.

The diagonal matrix is $D = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

and $P = \begin{pmatrix} 1 & 3 & 1 & -1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Example

Consider the matrix $A = \begin{pmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{pmatrix}$.

The characteristic polynomial of the matrix A is

$$q_A(\lambda) = \begin{vmatrix} 2-\lambda & 2 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -2 & 2-\lambda \end{vmatrix} = -(\lambda-1)^2(\lambda-5).$$

$E_1 = \langle (1,0,1), (-2,1,0) \rangle$, $E_5 = \langle (1,1,-1) \rangle$.
Then the matrix A is diagonalizable.
The diagonal matrix is $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$

Example

Consider the matrix $A = \begin{pmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{pmatrix}$.

The characteristic polynomial of the matrix A is

$$q_A(\lambda) = \begin{vmatrix} 7-\lambda & 4 & 16 \\ 2 & 5-\lambda & 8 \\ -2 & -2 & -5-\lambda \end{vmatrix} = -(\lambda-3)^2(\lambda-1).$$

$E_3 = \langle (1,-1,0),(4,0,-1) \rangle$, $E_1 = \langle (2,1,-1) \rangle$.

Then the matrix A is diagonalizable.

The diagonal matrix is $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ and the matrix P is $P = \begin{pmatrix} 2 & 1 & 4 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$

Example

Consider the matrix $A = \begin{pmatrix} 2 & -1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 3 \end{pmatrix}$. The characteristic polynomial of the matrix A is

$$q_A(\lambda) = \begin{vmatrix} 2-\lambda & -1 & 0 & \frac{1}{2} \\ 0 & 1-\lambda & 0 & \frac{1}{2} \\ -1 & 1 & 1-\lambda & -1 \\ 1 & -1 & 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)^3.$$

The matrix is diagonalizable if and only if the dimension the vector space E_2 is 3.
$E_2 = \langle(-1, 1, 0, 2), (-1, 0, 1, 0)\rangle$. Then the matrix is not diagonalizable.