Linear Transformations

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Definition of Linear Transformation

Definition

Let V and W be two vector spaces. A function $T\colon V \longrightarrow W$ is called a linear transformation from V to W if for all $u, v \in V$, $k \in \mathbb{R}$

1. $T(u+v) = T(u) + T(v)$, (additivity).
2. $T(ku) = kT(u)$, (homogeneity).

If $V = W$, we call a linear transformation from V to V a linear operator.

(Basic properties of linear transformations)

If $T\colon V \longrightarrow W$ is a linear transformation then

1. $T(0) = 0$.
2. $T(-u) = -T(u)$.
3. $T(u-v) = T(u) - T(v)$.

Example

1. $T : M_{m,n}(\mathbb{R}) \longrightarrow M_{n,m}(\mathbb{R})$ defined by $T(A) = A^T$;
2. $T : \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathcal{P}_3(\mathbb{R})$ defined by $T(a+bX+cX^2) = (a+b-2c)+cX+(a+c)X^2+(a+b)X^3$;
3. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by $T(x,y) = (x+3y, 2x-y, x+5y)$;
4. $T : \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathbb{R}^3$ defined by $T(a+bX+cX^2) = (a+3b-c, b-c, 2a-b+3c)$;

Example

The following functions are not linear transformations

1. $T\colon \mathbb{R}^3 \to \mathbb{R}^2$, defined by $T(x,y,z) = (xy, z)$ because $T(2,2,0) = (4,0) \neq 2T(1,1,0) = (2,0)$.
2. $T\colon \mathbb{R}^3 \to \mathbb{R}^2$, defined by $T(x,y,z) = (x+y-3z, z+y-1)$ because $T(0) \neq 0$;
3. $T\colon \mathbb{R}^3 \to \mathbb{R}^3$, defined by $T(x,y,z) = (x+y, z+y, x^2)$ because $T(2,0,0) = (2,0,4) \neq 2T(1,0,0) = (2,0,2)$.
4. $T\colon M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ defined by $T(A) = \det A$ because $\det(A+B) \neq \det A + \det B$ in general.

Theorem

If $T\colon V \longrightarrow W$ is a mapping, then T is a linear transformation if and only if

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v) \qquad \forall u, v \in V,\ \alpha, \beta \in \mathbb{R}.$$

Theorem

If $T\colon V \longrightarrow W$ is a linear transformation, then

$$T(\alpha_1 u_1 + \ldots + \alpha_n u_n) = \alpha_1 T(u_1) + \ldots + \alpha_n T(u_n),$$

for all $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $u_1, \ldots, u_n \in V$.

Remarks

1. If $S = \{u_1, \dots u_n\}$ is a basis of the vector space V. A linear transformation $T\colon V \longrightarrow W$ is well defined if $T(u_1), \dots, T(u_n)$ are defined.
2. The unique linear transformations $T\colon \mathbb{R} \longrightarrow \mathbb{R}$ are $T(x) = ax,\ a \in \mathbb{R}$.
3. The unique linear transformations $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ are $T(x, y) = ax + by,\ a, b \in \mathbb{R}$.

Theorem

If $A \in M_{m,n}(\mathbb{R})$, then the mapping $T_A \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined by: $T_A(X) = AX$ for all $X \in \mathbb{R}^n$ is a linear transformation and called the linear transformation associated to the matrix A.

Theorem

Let $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation and let $B = (e_1, \ldots, e_n)$ be a basis of the vector space $\mathbb{R}^n$ and $C = (u_1, \ldots, u_m)$ a basis of the vector space $\mathbb{R}^m$. Then $T = T_A$, where $A = \in M_{m,n}(\mathbb{R})$ with columns $[T(e_1)]_C, \ldots, [T(e_n)]_C$. The matrix A is called the matrix of the linear transformation T with respect to the basis B and C.

Theorem

Let V, W be two vector spaces and $S = \{v_1, \ldots, v_n\}$ a basis of the vector space V and $\{w_1, \ldots, w_n\}$ a set of vectors in the vector space W.
There is a unique linear transformation $T \colon V \longrightarrow W$ such that $T(v_j) = w_j$ for all $1 \leq j \leq n$.

Definition

Let $T\colon V \longrightarrow W$ be a linear transformation . The set $\{v \in V;\ T(v) = 0\}$ is called the kernel of the linear transformation T and denoted by: $\mathrm{ker(T)}$.
The set $\{T(v);\ v \in V\}$ is called the range or the image of the linear transformation T denoted by: $\mathrm{Im(T)}$.

Theorem

If $T\colon V \longrightarrow W$ is a linear transformation, then $\ker(\mathrm{T})$ is a vector sub-space of V and $\mathrm{Im}(\mathrm{T})$ is a vector sub-space of W.

Definition

If $T\colon V \longrightarrow W$ is a linear transformation then dimension the vector space $\ker(\mathrm{T})$ is called the nullity of the linear transformation T and denoted by: $(\mathrm{nullity}(T))$.
The dimension of the vector space $\mathrm{Im}(\mathrm{T})$ is called the rank of the linear transformation T and denoted by: $(\mathrm{rank}(T))$.

Example

If $A \in M_{m,n}(\mathbb{R})$ and $T_A \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ the linear transformation defined by: $T_A(X) = AX$, then $\mathrm{rank}(T) = \mathrm{rank} A$, and $\mathrm{Im}(T) = \mathrm{col} A$.

Example

Let $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the linear transformation defined by $T(x, y, z) = (2x - y + 3z, x - 2y + z)$.

$$(x, y, z) \in \ker(T) \iff \begin{cases} 2x - y + 3z = 0 \\ x - 2y + z = 0 \end{cases}$$

The extended matrix of this linear system is: $\left[\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ 1 & -2 & 1 & 0 \end{array}\right]$.

Then $(x, y, z) \in \ker(T) \iff x = 5y, z = -3y$. Hence $\ker(T) = \mathrm{Vect}\{(5, 1, -3)\}$.

As $T(x, y, z) = x(2, 1) + y(-1, -2) + z(3, 1)$, then

$$\mathrm{Im}(T) = \mathrm{Vect}\{(2, 1), (-1, -2), (3, 1)\} = \mathrm{Vect}\{(2, 1), (-1, -2)\}.$$

Theorem

If $T\colon V \longrightarrow W$ is a linear transformation and $\{v_1, \dots v_n\}$ is a basis of the vector space V, then the set $\{T(v_1), \dots T(v_n)\}$ generates the vector space $\mathrm{Im}(T)$.

The Dimension Theorem of the Linear Transformations

If $T: V \longrightarrow W$ is a linear transformation and if $\dim V = n$, then

$$\text{nullity}(T) + (\text{rank}(T) = n.$$

i.e.

$$\text{dimker}(T) + \text{dimIm}(T) = n.$$

Definition

If $T: V \longrightarrow W$ is a linear transformation,

1. T is called injective if $T(u) = T(v) \Rightarrow u = v$, for all $u, v \in V$.
2. T is called surjective if $\mathrm{Im}(T) = W$.

Theorem

If $T: V \longrightarrow W$ is a linear transformation. The linear transformation T is injective if and only if $\ker(T) = \{0\}$.

Corollary

If $T: V \longrightarrow W$ is a linear transformation and $\dim V = \dim W = n$. Then the linear transformation T is injective if and only if T is surjective.

Example

Consider the linear transformation $T\colon \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ defined by:

$$T(x,y,z,t) = (x-y, 2z+3t, y+4z+3t, x+6z+6t).$$

$(x,y,z,t) \in \mathrm{Ker}(T) \iff x = y = 3t = -2z$. Then $(6,6,-3,2)$ is a basis the kernel of T.

The range of T is the column space of the matrix

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 1 & 4 & 3 \\ 1 & 0 & 6 & 6 \end{pmatrix}.$$

The row reduced form of this matrix is

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $\{(1,0,0,1),(-1,0,1,0),(0,2,4,6)\}$ is a basis of the range of T.

Example

Let V, W be two vector spaces and $T\colon V \longrightarrow W$ a linear transformation. If T is injective and $S = \{u_1, \dots, u_n\}$ is a set of linearly independent, then the set $\{T(u_1), \dots, T(u_n)\}$ is linearly independent.

$$\begin{aligned} x_1 T(u_1) + \ldots + x_n T(u_n) = 0 \quad &\Longleftrightarrow \quad T(x_1 u_1 + \ldots + x_n u_n) = 0 \\ &\Longleftrightarrow \quad x_1 u_1 + \ldots + x_n u_n = 0 \end{aligned}$$

since T is injective and the set S is linearly independent, then $x_1 = \ldots = x_n = 0$.

Definition

Let $T\colon V \longrightarrow W$ be a linear transformation, $B = (u_1, \ldots, u_n)$ be a basis of V and $C = (v_1, \ldots, v_m)$ a basis of W. The matrix $[T]_B^C$ with columns $[T(u_1)]_C, \ldots, [T(u_n)]_C$ is called the matrix of the linear transformation T with respect to the basis B and the basis C. This matrix satisfies

$$[T(v)]_C = [T]_B^C \, [v]_B \, ; \qquad \forall v \in V.$$

If $V = W$ and $B = C$ we write the matrix $[T]_C$ instead of $[T]_B^C$.

Example

Let $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the linear transformation defined by the following: $T(x, y, z) = (2x - y + 3z, x - 2y + z)$. The matrix of T with respect to the standard basis of $\mathbb{R}^3$ is: $\begin{pmatrix} 2 & -1 & 3 \\ 1 & -2 & 1 \end{pmatrix}$

Example

Consider the linear transformation of $\mathbb{R}^3$ defined by: $T_1((1,0,0)) = (1,1,1)$, $T_1((0,1,0)) = (1,2,2)$, $T_1((0,0,1)) = (1,2,3)$. The matrix of T with respect to the standard basis of $\mathbb{R}^3$ is $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ and $T_1(x,y,z) = (x+y+z, x+2y+2z, x+2y+3z)$.

Theorem

If $T \colon V \longrightarrow V$ is a linear transformation and B and C are basis of the vector space V, then

$$[T]_B = {}_B P_C \, [T]_C \, {}_C P_B.$$

Example

Let $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation such that its matrix with respect to the standard basis C of the vector space $\mathbb{R}^3$ is

$$[T]_C = \begin{pmatrix} -3 & 2 & 2 \\ -5 & 4 & 2 \\ 1 & -1 & 1 \end{pmatrix}.$$

Consider the basis $B = \{u = (1,1,1),\ v = (1,1,0),\ w = (0,1,-1)\}$ of $\mathbb{R}^3$.

${}_C P_B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$. Then the matrix of T with respect to the basis S and the basis B is

$$ {}_B P_C = {}_S P_B^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 1 & 0 \end{pmatrix} $$

and

$$ [T]_B = {}_B P_C [T]_C \, {}_C P_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. $$

Example

Consider the linear transformation $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by: $T(x, y, z) = (3x + 2y, 3y + 2z, 9x - 4z)$.

The matrix of the linear transformation T is $A = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 3 & 2 \\ 9 & 0 & -4 \end{pmatrix}$.

The extended matrix of the linear system $AX = 0$ is: $\left[\begin{array}{ccc|c} 3 & 2 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 9 & 0 & -4 & 0 \end{array}\right]$.

This matrix is equivalent to the matrix $\left[\begin{array}{ccc|c} 3 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$.

Then $\ker(T) = \{0\}$ and the range of T is $\mathbb{R}^3$.

Consider the basis $S = \{(0,0,1),(0,1,1),(1,1,1)\}$ of rb^3.
The matrix the linear transformation T with respect to the basis S is $P^{-1}AP = \begin{pmatrix} -6 & -9 & 0 \\ 2 & 3 & 0 \\ 0 & 2 & 5 \end{pmatrix}$, with $P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Example

Consider $u_1 = \frac{1}{3}(1, 2, 2)$, $u_2 = \frac{1}{3}(2, 1, -2)$, $u_3 = \frac{1}{3}(2, -2, 1)$.
As the determinant $\begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{vmatrix} = -27$, then the set$\{u_1, u_2, u_3\}$ is a basis of $\mathbb{R}^3$. Also, we have $\|u_1\| = \|u_2\| = \|u_3\| = 1$ and $\langle u_1, u_2\rangle = \langle u_1, u_3\rangle = \langle u_2, u_3\rangle = 0$, then the set $\{u_1, u_2, u_3\}$ is an orthonormal basis of $\mathbb{R}^3$.

Define the linear transformation $T\colon \mathbb{R}^3 \mapsto \mathbb{R}^3$ by: $T(e_1) = u_1$, $T(e_2) = u_2$ and $T(e_3) = u_3$, where $\{e_1, e_2, e_3\}$ the standard basis of $\mathbb{R}^3$. The matrix of the linear transformation T with respect to the basis $\{e_1, e_2, e_3\}$ is

$$P = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

and

$$T(x, y, z) = \frac{1}{3}(x + 2y + 2z, 2x + y - 2z, 2x - 2y + z).$$

Define the linear transformation $S\colon \mathbb{R}^3 \mapsto \mathbb{R}^3$ by: $S(x,y,z) = (-x+2z, y+2z, 2x+2y)$. The matrix of S with respect to the basis $\{e_1, e_2, e_3\}$ is $A = \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 0 \end{pmatrix}$. The matrix of S with respect to basis $\{u_1, u_2, u_3\}$ is $B = P^{-1}AP$. As $P^{-1} = P^T = P$, hence

$$B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B^n = 3^n \begin{pmatrix} 1 & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A^n = PB^nP.$$

If $u = xu_1 + yu_2 + zu_3$, then $S(u) = 3xu_1 - 3u_2$.

Example

Let the matrix $A = \begin{pmatrix} 2 & -2 & 3 \\ -2 & 2 & 3 \\ 3 & 3 & -3 \end{pmatrix}$. We define the linear transformation $T\colon \mathbb{R}^3 \mapsto \mathbb{R}^3$ defined by the matrix A with respect to the standard basis (e_1, e_2, e_3) of the vector space $\mathbb{R}^3$.

1. Find $T(x, y, z)$.
2. Find an orthogonal basis (u_1, u_2, u_3) of the vector space $\mathbb{R}^3$ such that $T(u_1) = 3u_1$ and $T(u_2) = 4u_2$.
3. Find the matrix of the linear transformation T with respect to the basis (u_1, u_2, u_3).
4. We define the linear transformation $S\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by the following: $S(e_1) = u_1$, $S(e_2) = u_2$ and $S(e_3) = u_3$.
   Find the matrix P of the linear transformation S with respect to standard basis.

1. Prove that the matrix P has an inverse and find P^{-1}.
2. Let the linear transformation U defined by the matrix P^{-1} with respect to the standard basis.
   Find $U(u_k)$ for all $k = 1, 2, 3$.
3. Let $F = U \circ T \circ S$.
   Find $F(e_1)$, $F(e_2)$, $F(e_3)$.
   Find the matrix of the linear transformation F and conclude the value A^n for all $n \in \mathbb{N}$.

Solution

1. $$T(x, y, z) = (2x - 2y + 3z, -2x + 2y + 3z, 3x + 3y - 3z).$$
2. Let $u = (x, y, z)$.

$$T(u) = 3u \iff \begin{cases} -x - 2y + 3z = 0 \\ -2x - y + 3z \\ 3x + 3y - 6z = 0 \end{cases} \iff x = y = z.$$

We take $u_1 = (1, 1, 1)$.

$$T(u) = 4u \iff \begin{cases} -2x - 2y + 3z = 0 \\ -2x - 2y + 3z \\ 3x + 3y - 7z = 0 \end{cases} \iff \begin{cases} x = -y \\ z = 0 \end{cases}.$$

1. the matrix P has an inverse, then (u_1, u_2, u_3) is a basis .
   $P^{-1} = \frac{1}{6}\begin{pmatrix} 2 & 2 & 2 \\ 3 & -3 & 0 \\ 1 & 1 & -2 \end{pmatrix}$.
2. $U(u_1) = (1, 0, 0)$, $U(u_2) = (0, 1, 0)$, $U(u_3) = (0, 0, 1)$.
3. $F = U \circ T \circ S$.
   $F(e_1) = U \circ T(u_1) = 3U(u_1) = 3(1, 0, 0)$,
   $F(e_2) = U \circ T(u_2) = 4U(u_2) = 4(0, 1, 0)$,
   $F(e_3) = U \circ T(u_3) = -6U(u_3) = -6(0, 0, 1)$.
   The matrix of the linear transformation F is

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -6 \end{pmatrix}.$$

$$A^n = PD^nP^{-1}.$$