# Linear Transformations

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October 12, 2018

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# Definition of Linear Transformation

## Definition

Let $V$ and $W$ be two vector spaces and let $T\colon V \longrightarrow W$ be an application. We say that $T$ is a linear transformation If for all $u, v \in V$, $\alpha \in \mathbb{R}$

1. $T(u+v) = T(u) + T(v)$.
2. $T(\alpha u) = \alpha T(u)$.

# Remarks

If $T: V \longrightarrow W$ is a linear transformation then

1. $T(0) = 0.$
2. $T(-u) = -T(u).$
3. $T(u - v) = T(u) - T(v).$

# Example

Select from the following functions which is a linear transformation

$$
\begin{aligned}
&T_1 : \mathbb{R}^3 \to \mathbb{R}^2, && T_1(x, y, z) = (x + y + z, x - z + y)\\
&T_2 : \mathbb{R}^3 \to \mathbb{R}^2, && T_2(x, y, z) = (xy, z)\\
&T_3 : \mathbb{R}^3 \to \mathbb{R}^2, && T_3(x, y, z) = (x + y - 3z, z + y - 1)\\
&T_4 : \mathbb{R}^3 \to \mathbb{R}^3, && T_4(x, y, z) = (x + y, z + y, x^2)\\
&T_5 : \mathbb{R}^3 \to \mathbb{R}^3, && T_5(x, y, z) = (x + y, z + y, 0)\\
&T_6 : \mathbb{R}^3 \to \mathbb{R}^3, && T_6(x, y, z) = (-x + 2z, y + 2z, 2x + 2y)\\
&T_7 : \mathbb{R}^3 \to \mathbb{R}, && T_7(x, y, z) = x + y - z.
\end{aligned}
$$

# Solution

$T_1$ is a linear transformation .
$T_2$ is not a linear transformation
$T_3$ is not a linear transformation because $T(0) \neq 0$.
$T_4$ is not a linear transformation
$T_5$ is a linear transformation .
$T_6$ is a linear transformation .
$T_7$ is a linear transformation .

# Example

Let the vector space $V = M_n(\mathbb{R})$. We define the function $T: V \longrightarrow \mathbb{R}$ as follows: $T(A) = \det A$.
The function $T$ is not linear because $\det(A+B) \neq \det A + \det B$.

## Theorem

If $T\colon V \longrightarrow W$ is a mapping, then $T$ is a linear transformation if and only if

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v) \qquad \forall u, v \in V,\ \alpha, \beta \in \mathbb{R}.$$

# Remarks

1. If $T\colon V \longrightarrow W$ is a linear transformation, then $T(\alpha_1 u_1 + \ldots + \alpha_n u_n) = \alpha_1 T(u_1) + \ldots + \alpha_n T(u_n)$.
2. If $T\colon V \longrightarrow W$ is a linear transformation and $S = \{u_1, \ldots u_n\}$ is a basis of the vector space $V$.
   The linear transformation is well defined if $T(u_1), \ldots, T(u_n)$ are defined.
3. The unique linear transformations $T\colon \mathbb{R} \longrightarrow \mathbb{R}$ are $T(x) = ax$, $a \in \mathbb{R}$.
4. The unique linear transformations $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ are $T(x, y) = ax + by$, $a, b \in \mathbb{R}$.

**Theorem**

If $A \in M_{m,n}(\mathbb{R})$, then the mapping $T_A \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined by: $T_A(X) = AX$ for all $X \in \mathbb{R}^n$ is a linear transformation and called the linear transformation associated to the matrix $A$.

## Theorem

Let $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation and let $B = (e_1, \dots, e_n)$ be a basis of the vector space $\mathbb{R}^n$ and $C = (u_1, \dots, u_m)$ a basis of the vector space $\mathbb{R}^m$. Then $T = T_A$, where $A = [a_{i,j}] \in M_{m,n}(\mathbb{R})$ and its columns are in order $[T(e_1)]_C, \dots, [T(e_n)]_C$.
The matrix $A$ is called the matrix of the linear transformation $T$ with respect to the basis $B$ and $C$.

## Theorem

Let $V, W$ be two vector spaces and $S = \{v_1, \ldots, v_n\}$ a basis of the vector space $V$ and $\{w_1, \ldots, w_n\}$ a set of vectors in the vector space $W$.
There is a unique linear transformation $T \colon V \longrightarrow W$ such that $T(v_j) = w_j$ for all $1 \leq j \leq n$.

**Definition**

Let $T\colon V \longrightarrow W$ be a linear transformation . The set $\{v \in V;\ T(v) = 0\}$ is called the kernel of the linear transformation $T$ and denoted by: $\mathrm{ker(T)}$.
The set $\{T(v);\ v \in V\}$ is called the image of the linear transformation $T$ denoted by: $\mathrm{Im(T)}$.

**Theorem**

If $T\colon V \longrightarrow W$ is a linear transformation, then $\ker(\mathrm{T})$ is a vector sub-space of $V$ and $\mathrm{Im}(\mathrm{T})$ is a vector sub-space of $W$.

**Definition**

If $T : V \longrightarrow W$ is a linear transformation then dimension the vector space $\ker(\mathrm{T})$ is called the nullity of the linear transformation $T$ and denoted by: $(\mathrm{nullity}(T))$.
The dimension of the vector space $\mathrm{Im}(\mathrm{T})$ is called the rank of the linear transformation $T$ and denoted by: $(\mathrm{rank}(T))$.

## Example

If $A \in M_{m,n}(\mathbb{R})$ and $T_A\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ the linear transformation defined by: $T_A(X) = AX$, then $\mathrm{rank}(T) = \mathrm{rank}A$, and $\mathrm{Im}(T) = \mathrm{col}A$.

# Example

Let $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the linear transformation defined by the following:
$T(x,y,z) = (2x - y + 3z, x - 2y + z)$.
$(x,y,z) \in \ker(T) \iff \begin{cases} 2x - y + 3z = 0 \\ x - 2y + z = 0 \end{cases}$

The extended matrix of this linear system is: $\left[\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ 1 & -2 & 1 & 0 \end{array}\right]$.

Then

$$(x,y,z) \in \ker(T) \iff x = 5y, z = -3y.$$

$$\ker(T) = \langle (5,1,-3) \rangle.$$

$T(x,y,z) = x(2,1) + y(-1,-2) + z(3,1)$.
Then

$$\mathrm{Im}(T) = \langle (2,1), (-1,-2), (3,1) \rangle = \langle (2,1), (-1,-2) \rangle.$$

**Theorem**

If $T\colon V \longrightarrow W$ is a linear transformation and $\{v_1, \dots v_n\}$ is a basis of the vector space $V$, then the set $\{T(v_1), \dots T(v_n)\}$ generates the vector space $\mathrm{Im}(T)$.

### The Dimension Theorem of the Linear Transformations

If $T\colon V \longrightarrow W$ is a linear transformation and if $\dim V = n$, then

$$\mathrm{nullity}(T) + (\mathrm{rank}(T) = n.$$

i.e.

$$\mathrm{dimker}(T) + \mathrm{dimIm}(T) = n.$$

## Definition

If $T\colon V \longrightarrow W$ is a linear transformation,

1. We say that $T$ is injective if for all $u, v \in V$,
$$T(u) = T(v) \Rightarrow u = v.$$
2. We say that $T$ is surjective if $\mathrm{Im}(T) = W$.

**Theorem**

If $T: V \longrightarrow W$ is a linear transformation. The linear transformation $T$ is injective if and only if $\ker(T) = \{0\}$.

**Corollary**

If $T\colon V \longrightarrow W$ is a linear transformation and $\dim V = \dim W = n$. Then the linear transformation $T$ is injective if and only if $T$ is surjective.

# Example

Give a basis of the image of and of the kernel of the following linear transformation $T: \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ defined by following:

$$T(x, y, z, t) = (x - y, 2z + 3t, y + 4z + 3t, x + 6z + 6t).$$

# Solution

$(x, y, z, t) \in \mathrm{Ker}(T) \iff x = y = 3t = -2z.$
Then $(6, 6, -3, 2)$ is a basis the kernel of the linear transformation.
The image of the linear transformation $T$ is spanned by columns of the following matrix

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 1 & 4 & 3 \\ 1 & 0 & 6 & 6 \end{pmatrix}$$

and the matrix

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is a row reduced form of this matrix. Then

$$\{(1,0,0,1),(-1,0,1,0),(0,2,4,6)\}$$

is a basis of the image of the linear transformation.

## Example

Let $V, W$ be two vector spaces and $T\colon V \longrightarrow W$ a linear transformation.
If the linear transformation injective and $S = \{u_1, \dots, u_n\}$ is a set of linearly independent vectors, then the set $\{T(u_1), \dots, T(u_n)\}$ is a set of linearly independent vectors.

# Solution

$$\begin{aligned} a_1 T(u_1) + \ldots + a_n T(u_n) = 0 &\iff T(a_1 u_1 + \ldots + a_n u_n) = 0 \\ &\iff a_1 u_1 + \ldots + a_n u_n = 0 \end{aligned}$$

since $T$ is injective and since the set $S$ is linearly independent, then $a_1 = \ldots = a_n = 0$.

**Theorem**

Let $T\colon V \longrightarrow W$ be a linear transformation and let $B = (u_1, \ldots, u_n)$ be a basis of the vector space $V$ and $C = (v_1, \ldots, v_m)$ basis of the vector space $W$. Then there is a unique matrix $[T]_B^C$ such that its columns $[T(u_1)]_C, \ldots, [T(u_n)]_C$. The matrix $[T]_B^C$ is called the matrix of the linear transformation $T$ with respect to the basis $B$ and the basis $C$. and satisfies

$$[T(v)]_C = [T]_B^C [v]_B ; \qquad \forall v \in V.$$

If $V = W$ and $B = C$ we write the matrix $[T]_C$ instead of $[T]_B^C$.

# Example

Let $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the linear transformation defined by the following:
$T(x, y, z) = (2x - y + 3z, x - 2y + z)$.
The matrix of the linear transformation $T$ with respect to the standard basis of the vector space $\mathbb{R}^3$ is: $\begin{pmatrix} 2 & -1 & 3 \\ 1 & -2 & 1 \end{pmatrix}$

## Example

Find the matrix of the linear transformation with respect to the standard basis of the vector space $\mathbb{R}^3$ and find $T_j(x, y, z)$ if

1. $T_1((1,0,0)) = (1,1,1)$, $T_1((0,1,0)) = (1,2,2)$, $T_1((0,0,1)) = (1,2,3)$
2. $T_2((1,0,0)) = (1,-1,1)$, $T_2((0,1,0)) = (-1,1,1)$, $T_2((0,0,1)) = (-1,-1,1)$
3. $T_3((1,0,0)) = (1,1,1)$, $T_3((0,1,0)) = (1,2,1)$, $T_3((0,0,1)) = (2,-2,1)$.

# Solution

1. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$,
   $T_1(x, y, z) = (x + y + z, x + 2y + 2z, x + 2y + 3z)$.
2. $\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$,
   $T_2(x, y, z) = (x - y - z, -x + y - z, x + y + z)$.
3. $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & -2 \\ 1 & 1 & 1 \end{pmatrix}$,
   $T_3(x, y, z) = (x + y + 2z, x + 2y - 2z, x + y + z)$.

**Theorem**

If $T: V \longrightarrow V$ is a linear transformation and $B$ and $C$ are basis of the vector space $V$, then

$$[T]_B = {}_BP_C\,[T]_C\,{}_CP_B.$$

# Example

Let $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation such that its matrix with respect to the standard basis $C$ of the vector space $\mathbb{R}^3$ is

$$[T]_C = \begin{pmatrix} -3 & 2 & 2 \\ -5 & 4 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

Find the matrix of the linear transformation $[T]_B$ with respect to the following basis $B$

$$B = \{u = (1,1,1),\ v = (1,1,0),\ w = (0,1,-1)\}.$$

# Solution

The matrix of the linear transformation with respect to the basis $B$ and $C$ is

$$_CP_B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

Then the matrix of the linear transformation with respect to the basis $S$ and the basis $B$ is

$$_BP_C = {}_SP_B^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

and

$$[T]_B = {}_BP_C[T]_C\,{}_CP_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

# Example

Let the linear transformation $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by the following:

$$T(x, y, z) = (3x + 2y, 3y + 2z, 9x - 4z).$$

1. Give the matrix of the linear transformation $T$.
2. Give the kernel of and image of the linear transformation $T$.
3. Find the matrix the linear transformation $T$ with respect to the basis $S = \{(0, 0, 1), (0, 1, 1), (1, 1, 1)\}$.

# Solution

1. The matrix of the linear transformation $T$ is
$$A = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 3 & 2 \\ 9 & 0 & -4 \end{pmatrix}$$
2. The extended matrix of the linear system $AX = 0$ is:
$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 9 & 0 & -4 & 0 \end{array}\right].$$ This matrix is equivalent to the matrix
$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$
Then $\ker(T) = \{0\}$ and the image of the linear transformation $T$is: $\mathbb{R}^3$.
3. Let $P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

# Example

Let $u_1 = \frac{1}{3}(1, 2, 2)$, $u_2 = \frac{1}{3}(2, 1, -2)$, $u_3 = \frac{1}{3}(2, -2, 1)$.

1. Prove that $\{u_1, u_2, u_3\}$ is an orthonormal basis of the vector space $\mathbb{R}^3$.
2. We define the linear transformation $T\colon \mathbb{R}^3 \mapsto \mathbb{R}^3$ by the following:
   $T(e_1) = u_1$, $T(e_2) = u_2$ and $T(e_3) = u_3$, where $\{e_1, e_2, e_3\}$ the standard basis of the vector space $\mathbb{R}^3$.
   Find $P$ the matrix of the linear transformation $T$ with respect to the basis $\{e_1, e_2, e_3\}$ and find $T(x, y, z)$.
3. We define the linear transformation $S\colon \mathbb{R}^3 \mapsto \mathbb{R}^3$ by the following:
   $S(x, y, z) = (-x + 2z, y + 2z, 2x + 2y)$.
   Prove that $S$ is a linear transformation and find its matrix $A$ with respect to the basis $\{e_1, e_2, e_3\}$.

# Solution

1. As the determinant
$$\begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{vmatrix} = -27$$
then $\{u_1, u_2, u_3\}$ is a basis and as $\|u_1\| = \|u_2\| = \|u_3\| = 1$ and $\langle u_1, u_2\rangle = \langle u_1, u_3\rangle = \langle u_2, u_3\rangle = 0$, then $\{u_1, u_2, u_3\}$ is an orthonormal basis of the vector space $\mathbb{R}^3$.

2. $$P = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$
and

# Example

Let the matrix $A = \begin{pmatrix} 2 & -2 & 3 \\ -2 & 2 & 3 \\ 3 & 3 & -3 \end{pmatrix}$. We define the linear transformation $T\colon \mathbb{R}^3 \mapsto \mathbb{R}^3$ defined by the matrix $A$ with respect to the standard basis $(e_1, e_2, e_3)$ of the vector space $\mathbb{R}^3$.

1. Find $T(x, y, z)$.
2. Find an orthogonal basis $(u_1, u_2, u_3)$ of the vector space $\mathbb{R}^3$ such that $T(u_1) = 3u_1$ and $T(u_2) = 4u_2$.
3. Find the matrix of the linear transformation $T$ with respect to the basis $(u_1, u_2, u_3)$.
4. We define the linear transformation $S\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by the following: $S(e_1) = u_1$, $S(e_2) = u_2$ and $S(e_3) = u_3$.
   Find the matrix $P$ of the linear transformation $S$ with respect to standard basis.

1. Prove that the matrix $P$ has an inverse and find $P^{-1}$.
2. Let the linear transformation $U$ defined by the matrix $P^{-1}$ with respect to the standard basis.
   Find $U(u_k)$ for all $k = 1, 2, 3$.
3. Let $F = U \circ T \circ S$.
   Find $F(e_1)$, $F(e_2)$, $F(e_3)$.
   Find the matrix of the linear transformation $F$ and conclude the value $A^n$ for all $n \in \mathbb{N}$.

# Solution

1. $$T(x, y, z) = (2x - 2y + 3z, -2x + 2y + 3z, 3x + 3y - 3z).$$

2. Let $u = (x, y, z)$.

$$T(u) = 3u \iff \begin{cases} -x - 2y + 3z = 0 \\ -2x - y + 3z \\ 3x + 3y - 6z = 0 \end{cases} \iff x = y = z.$$

We take $u_1 = (1, 1, 1)$.

$$T(u) = 4u \iff \begin{cases} -2x - 2y + 3z = 0 \\ -2x - 2y + 3z \\ 3x + 3y - 7z = 0 \end{cases} \iff \begin{cases} x = -y \\ z = 0 \end{cases}.$$

1. the matrix $P$ has an inverse, then $(u_1, u_2, u_3)$ is a basis .
$P^{-1} = \frac{1}{6}\begin{pmatrix} 2 & 2 & 2 \\ 3 & -3 & 0 \\ 1 & 1 & -2 \end{pmatrix}$.
2. $U(u_1) = (1, 0, 0)$, $U(u_2) = (0, 1, 0)$, $U(u_3) = (0, 0, 1)$.
3. $F = U \circ T \circ S$.
$F(e_1) = U \circ T(u_1) = 3U(u_1) = 3(1, 0, 0)$,
$F(e_2) = U \circ T(u_2) = 4U(u_2) = 4(0, 1, 0)$,
$F(e_3) = U \circ T(u_3) = -6U(u_3) = -6(0, 0, 1)$.
The matrix of the linear transformation $F$ is

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -6 \end{pmatrix}.$$

$$A^n = PD^nP^{-1}.$$