

Inner Product Spaces and Orthogonality

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Inner Product

Definition

An inner product on a vector space V is a function that maps each ordered pair of vectors x, y to a real $\langle x, y \rangle$ such that the following four properties hold.

Let V be a vector space on \mathbb{R} . A function $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{R}$ is called an inner product on V if the following five properties hold:

- 1 For all $u, v \in V$: $\langle u, v \rangle = \langle v, u \rangle$;
- 2 For all $u, v, w \in V$: $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$;
- 3 For all $u, v \in V$ and $\alpha \in \mathbb{R}$: $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$;
- 4 For all $u \in V$: $\langle u, u \rangle \geq 0$;
- 5 $\langle u, u \rangle = 0 \iff u = 0$.

Examples

- 1 The standard inner product on \mathbb{R}^n defined by:

$$\langle u, v \rangle = \sum_{j=1}^n x_j y_j = x_1 y_1 + \dots + x_n y_n,$$

where $u = (x_1, \dots, x_n)$ and $v = (y_1, \dots, y_n)$.

- 2 If $E = \mathcal{C}([0, 1])$ the vector space of continuous functions on $[0, 1]$. For all $f, g \in E$, we define the inner product of f and g by:

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt.$$

Remarks

If $(E, \langle \cdot, \cdot \rangle)$ is an inner product space and $u, v, w, x \in E$,
 $a, b, c, d \in \mathbb{R}$, we have:

$$\langle u + v, w + x \rangle = \langle u, w \rangle + \langle u, x \rangle + \langle v, w \rangle + \langle v, x \rangle.$$

$$\begin{aligned} \langle au + bv, cw + dx \rangle &= ac\langle u, w \rangle + ad\langle u, x \rangle \\ &\quad + bc\langle v, w \rangle + bd\langle v, x \rangle. \end{aligned}$$

Example

On \mathbb{R}^2 , define for $u = (x, y)$ and $v = (x', y')$, the function:

$$\langle u, v \rangle = 2xx' + yy' - xy' - yx'.$$

$\langle \cdot, \cdot \rangle$ is an inner product on \mathbb{R}^2 . We prove that $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0 \iff u = 0$. The other properties are left to the reader.

$$\langle u, u \rangle = 2x^2 + y^2 - 2xy = (x - y)^2 + x^2 \geq 0 \text{ and}$$

$$\langle u, u \rangle = 0 \iff u = 0.$$

Example

On \mathbb{R}^3 , define for $u = (x, y, z)$ and $v = (x', y', z')$ the function:

$$\langle u, v \rangle = 2xx' + yy' + 3zz' - xy' - yx' + yz' + zy'.$$

$\langle \cdot, \cdot \rangle$ is an inner product on \mathbb{R}^3 . We prove that $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0 \iff u = 0$. The other properties are left to the reader.

$$\begin{aligned} \langle u, u \rangle &= (y + z - x)^2 - (z - x)^2 + 2x^2 + 3z^2 \\ &= (y + z - x)^2 + (x + z)^2 + z^2 \geq 0 \end{aligned}$$

$$\langle u, u \rangle = 0 \iff z = x = y = 0 \iff u = 0.$$

Example

On \mathbb{R}^3 , define for $u = (x, y, z)$ and $v = (x', y', z')$ the function:

$$\langle u, v \rangle = 2xx' + yy' + zz' - xy' - yx' + yz' + zz'.$$

\langle , \rangle is not an inner product on \mathbb{R}^3 .

$$\begin{aligned} \langle u, u \rangle &= (y + z - x)^2 - (z - x)^2 + 2x^2 + z^2 \\ &= (y + z - x)^2 + x^2 + 2xz \\ &= (y + z - x)^2 + (x + z)^2 - z^2. \end{aligned}$$

Example

If $A = (a_{j,k}) \in M_n(\mathbb{R})$, we define the trace of the matrix A by:

$$\operatorname{tr}(A) = \sum_{j=1}^n a_{j,j}.$$

The function $(A, B) \mapsto \langle A, B \rangle = \operatorname{tr}(AB^T)$ is an inner product on the vector space $M_n(\mathbb{R})$.

Exercise

If $u = (x, y, z)$, $v = (x', y', z')$, we define the following functions:
 $f, g, h, k: \mathbb{R}^2 \times \mathbb{R}^3 \rightarrow \mathbb{R}$.

- 1 $f(u, v) = xx' + yy' + 2zz' + yx' + 2xy' + yz' + y'z$.
- 2 $g(u, v) = xy' + yx' + yz' + zy' + 3xz' + 3zx'$.
- 3 $h(u, v) = xx' + yy' + zz' + yx' + xy' + yz' + y'z + zx' + xz'$.
- 4 $k(u, v) = xx' + yy' + zz' - yz' - zy' + xz' + x'z$.

Select from which the functions f, g, h, k is an inner product on \mathbb{R}^3 .

Solution

- ① $f(u, v) - f(v, u) = xy' - yx'$. Then f is not an inner product on \mathbb{R}^3 .
- ② $g(u, u) = 2xy + 2yz + 6xz = 2(x + z)(y + 3z) - 6z^2 = (x + y + 4z)^2 - (x - y - 2z)^2 - 6z^2$.
Then g is not an inner product on \mathbb{R}^3 .

③

$$\begin{aligned} h(u, u) &= x^2 + y^2 + z^2 + 2xy + 2yz + 2xz \\ &= (x + y + z)^2 \end{aligned}$$

Then h is not an inner product on \mathbb{R}^3 because

$$h(u, u) = 0 \not\Rightarrow u = 0.$$

④

$$\begin{aligned}k(u, u) &= x^2 + y^2 + z^2 - 2yz + 2xz \\ &= (x + z)^2 + y^2 - 2yz \\ &= (x + z)^2 + (y - z)^2 - z^2\end{aligned}$$

Then k is not an inner product on \mathbb{R}^3 because

$$k(u, u) = 0 \not\Rightarrow u = 0.$$

Example

Find the values of a, b such that

$$\langle (x, y), (x', y') \rangle = xx' + yy' + axy' + byx'$$

is an inner product on \mathbb{R}^2 .

Solution

$$\langle (x, y), (x', y') \rangle = \langle (x', y'), (x, y) \rangle \text{ if } a = b.$$

$$\begin{aligned}\langle (x, y), (x, y) \rangle &= x^2 + y^2 + 2axy \\ &= (x + ay)^2 + y^2(1 - a^2).\end{aligned}$$

Then $\langle \cdot, \cdot \rangle$ is an inner product on \mathbb{R}^2 if and only if $|a| < 1$.

Definition

Let $(E, \langle \cdot, \cdot \rangle)$ be an inner product space.

- ① If $u \in E$, we define the norm of the vector u by:

$$\|u\| = \sqrt{\langle u, u \rangle}.$$

- ② If $u, v \in E$, we define distance between u and v by:

$$d(u, v) = \|u - v\|.$$

- ③ We define the angle $0 \leq \theta \leq \pi$ between the vectors $u, v \in E$ by:

$$\cos \theta = \frac{\langle u, v \rangle}{\|u\| \cdot \|v\|}$$

Let the inner product space $M_2(\mathbb{R}), \langle \cdot, \cdot \rangle$ defined by:

$$\langle A, B \rangle = \text{tr}(AB^T).$$

Find $\cos \theta$ If θ is the angle between the matrices

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

$$AB^T = \begin{pmatrix} 1 & 0 \\ 7 & 5 \end{pmatrix}, \|A\|^2 = 15, \|B\|^2 = 7.$$

Then

$$\cos \theta = \frac{2\sqrt{3}}{\sqrt{35}}.$$

Theorem (Cauchy-Schwarz Inequality)

If $(E, \langle \cdot, \cdot \rangle)$ is an inner product space and $u, v \in E$,
then

$$|\langle u, v \rangle| \leq \|u\| \|v\|. \quad (1)$$

We have the equality in (1) if the vectors u, v are linearly dependent.

Proof

Let $Q(t)$ be the polynomial

$$Q(t) = \|u + tv\|^2 = \|u\|^2 + 2t\langle u, v \rangle + t^2\|v\|^2.$$

Since $Q(t) \geq 0$ for all $t \in \mathbb{R}$, then the discriminant of $Q(t)$ is non positive. Then

$$\langle u, v \rangle^2 \leq \|u\|^2\|v\|^2.$$

If $|\langle u, v \rangle| = \|u\|\|v\|$, this mean that the discriminant of $Q(t)$ is zero. Then the equation $Q(t) = 0$ has a solution. This means that the vectors u, v are linearly dependent.

Theorem

If $(E, \langle \cdot, \cdot \rangle)$ is an inner product space and $u, v \in E$, then

$$\|u + v\| \leq \|u\| + \|v\|.$$

Proof

$$\begin{aligned} \|u + v\|^2 &= \|u\|^2 + \|v\|^2 + 2\langle u, v \rangle \\ &\leq \|u\|^2 + \|v\|^2 + 2\|u\| \|v\| = (\|u\| + \|v\|)^2. \end{aligned}$$

Definition

If $(E, \langle \cdot, \cdot \rangle)$ is an inner product space. We say that the vectors $u, v \in E$ are orthogonal and we denote $u \perp v$ if $\langle u, v \rangle = 0$.

Theorem (Pythagor's Theorem)

If $u \perp v$ if and only if

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

Proof

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2 + 2\langle u, v \rangle = \|u\|^2 + \|v\|^2.$$

Definition

If $(E, \langle \cdot, \cdot \rangle)$ is an inner product space. We say that set $S = \{e_1, \dots, e_n\}$ of non zeros vectors is orthogonal if

$$\langle e_j, e_k \rangle = 0, \quad \forall 1 \leq j \neq k \leq n.$$

and we say that S is normal if

$$\|e_j\| = 1, \quad \forall 1 \leq j \leq n.$$

and we say that it is orthonormal if

$$\langle e_j, e_k \rangle = \delta_{j,k}, \quad \forall 1 \leq j, k \leq n.$$

($\delta_{j,k} = 0$ If $j \neq k$ and $\delta_{j,j} = 1$.)

Theorem

Any set of non zero orthogonal vectors is linearly independent .

Theorem

If $(E, \langle \cdot, \cdot \rangle)$ is an inner product space and if $S = \{e_1, \dots, e_n\}$ is an orthonormal basis of E , then for all $u \in E$

$$u = \langle u, e_1 \rangle e_1 + \dots + \langle u, e_n \rangle e_n.$$

Proof

If $u = \sum_{j=1}^n a_j e_j$, then $\langle u, e_k \rangle = \sum_{j=1}^n a_j \langle e_j, e_k \rangle = a_k$.

Theorem

(Gramm-Schmidt Algorithm) If $(E, \langle \cdot, \cdot \rangle)$ is an inner product space and (v_1, \dots, v_n) a set of linearly independent vectors in E , there is a unique orthonormal set (e_1, \dots, e_n) such that

- 1 for all $k \in \{1, \dots, n\}$,

$$\text{Vect}(e_1, \dots, e_k) = \text{Vect}(v_1, \dots, v_k),$$

- 2 for all $k \in \{1, \dots, n\}$,

$$\langle e_k, v_k \rangle > 0.$$

Proof

We construct in the first time an orthogonal set (u_1, \dots, u_n) such that:

$$\begin{cases} u_1 &= v_1 \\ u_2 &= v_2 - \frac{\langle u_1, v_2 \rangle}{\|u_1\|^2} u_1 \\ &\vdots \\ u_n &= v_n - \sum_{i=1}^{n-1} \frac{\langle u_i, v_n \rangle}{\|u_i\|^2} u_i. \end{cases}$$

We construct the set (e_1, \dots, e_n) from (u_1, \dots, u_n) as follows:

$$e_k = \frac{u_k}{\|u_k\|}, \quad k \in \{1, \dots, n\}.$$

Example

Let F be the vector sub-space of \mathbb{R}^4 spanned by the vectors $S = \{u = (1, 1, 0, 0), v = (1, 0, -1, 0), w = (0, 0, 1, 1)\}$.

- 1 Prove that S is a basis of the sub-space F .
- 2 In use of Gram-Schmidt Algorithm, find an orthonormal basis of F , (with respect to the Euclidean inner product).

Solution

- ① Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ with columns the vectors u, v, w .

The matrix $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is a row reduced form of the

matrix A . This proves that S is a basis of the sub-space F .

② $u_1 = \frac{1}{\sqrt{2}}(1, 1, 0, 0), u_2 = \frac{1}{\sqrt{6}}(1, -1, -2, 0),$

$u_3 = \frac{1}{\sqrt{12}}(1, -1, 1, 3).$

$\{u_1, u_2, u_3\}$ is an orthonormal basis of the sub-space F .

Exercise

Consider the base $\{u_1 = (1, -1), u_2 = (1, 2)\}$ of \mathbb{R}^2 and the following inner product

$$\langle (x, y), (x', y') \rangle = xx' + xy' + yx' + 2yy'.$$

The vector u_1 is unitary with respect to the inner product and using the Gram-Schmidt algorithm, the vector $v_2 = (1, 0)$ is orthogonal to u_1 and $\{v_1 = (1, -1), v_2 = (1, 0)\}$ is an orthonormal basis of \mathbb{R}^2 .

Example

Consider the basis $S = \{u_1, u_2, u_3, u_4\}$ of the space $M_2(\mathbb{R})$ defined by: $u_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $u_3 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $u_4 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

Using Gram-Schmidt algorithm to construct an orthonormal basis from the basis S , we find: $v_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

$\langle u_2, v_1 \rangle = \frac{2}{\sqrt{3}}$, $u_2 - \langle u_2, v_1 \rangle v_1 = \frac{1}{3} \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$. Then

$v_2 = \frac{1}{\sqrt{15}} \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$.

$$\langle u_3, v_1 \rangle = \sqrt{3}, \quad \langle u_3, v_2 \rangle = \frac{3}{\sqrt{15}}$$

$$u_3 - \langle u_3, v_1 \rangle v_1 - \langle u_3, v_2 \rangle v_2 = \frac{1}{5} \begin{pmatrix} -1 & 3 \\ -3 & 4 \end{pmatrix}. \text{ Then}$$

$$v_3 = \frac{1}{\sqrt{35}} \begin{pmatrix} -1 & 3 \\ -3 & 4 \end{pmatrix}.$$

$$\langle u_4, v_1 \rangle = 0, \quad \langle u_4, v_2 \rangle = \frac{6}{\sqrt{15}}, \quad \langle u_4, v_3 \rangle = \frac{4}{\sqrt{35}}$$

$$u_4 - \langle u_4, v_1 \rangle v_1 - \langle u_4, v_2 \rangle v_2 - \langle u_4, v_3 \rangle v_3 = \frac{1}{35} \begin{pmatrix} -10 & -39 \\ -29 & -29 \end{pmatrix}. \text{ Then}$$

$$v_4 = \frac{1}{\sqrt{7}} \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}.$$

Exercise

Let F be the vector sub-space of the Euclidean space \mathbb{R}^4 spanned by the following vectors $u_1 = (1, 2, 0, 2)$, $u_2 = (-1, 1, 1, 1)$.

- 1 Use Gram-Schmidt algorithm to construct an orthonormal basis of the vector sub-space F
- 2 Prove that the set $F^\perp = \{u \in \mathbb{R}^4 : \langle u, v \rangle = 0, \forall v \in F\}$ is a vector sub-space of \mathbb{R}^4 .
- 3 Find an orthonormal basis of F^\perp .

Solution

- ① $v_1 = \frac{1}{3}u_1$, $\langle u_2, v_1 \rangle = 1$,
 $u_2 - \langle u_2, v_1 \rangle v_1 = (0, 3, 1, -1) - \frac{1}{3}(-1, 1, 1, 1) = \frac{1}{3}(-4, 1, 3, 1)$.
 Then $v_2 = \frac{1}{3\sqrt{3}}(-4, 1, 3, 1)$.
 (v_1, v_2) is an orthonormal basis of the vector sub-space F .
- ② If $v_1, v_2 \in F^\perp$, $\alpha, \beta \in \mathbb{R}$ and $u \in F$, then

$$\langle \alpha v_1 + \beta v_2, u \rangle = \alpha \langle v_1, u \rangle + \beta \langle v_2, u \rangle = 0.$$

Then F^\perp is a vector sub-space of \mathbb{R}^4 .

- ③ Let $u = (x, y, z, t) \in \mathbb{R}^4$.

$$u \in F^\perp \iff \begin{cases} \langle u, u_1 \rangle = 0 \\ \langle u, u_2 \rangle = 0 \end{cases} \iff \begin{cases} x + 2y + 2t = 0 \\ -x + y + z + t = 0 \end{cases}$$

$$\begin{cases} x + 2y + 2t = 0 \\ -x + y + z + t = 0 \end{cases} \iff \begin{cases} x = \frac{2}{3}z \\ y = -\frac{z}{3} - t \end{cases}$$

Then $u \in F^\perp \iff u = -\frac{z}{3}(-2, 1, -3, 0) + t(0, -1, 0, 1)$.

The vectors $e_1 = (-2, 1, -3, 0)$, $e_2 = (0, -1, 0, 1)$ is an orthogonal basis of the vector sub-space F^\perp .

$$w_1 = \frac{1}{\sqrt{14}}e_1, \langle w_1, e_2 \rangle = -\frac{1}{\sqrt{14}},$$

$$e_2 - \langle e_2, w_1 \rangle w_1 = \frac{1}{14}(2, 13, 3, 14).$$

Then $(\frac{1}{\sqrt{14}}(-2, 1, -3, 0), \frac{1}{3\sqrt{42}}(2, 13, 3, 14))$ is an orthonormal basis of the vector sub-space F^\perp .