Local Cauchy's Theory

BLEL Mongi

Department of Mathematics
King Saud University

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Definition

Let Ω be an open subset of $\mathbb{C}$ and I a closed interval in $\mathbb{R}$.

1. *A continuous function $\gamma\colon I \longrightarrow \Omega$ is called a curve or a path in Ω.*
2. *A continuously differentiable function $\gamma\colon [a,b] \longrightarrow \Omega$ is called a differentiable path or a differentiable curve in Ω, and we say that γ is parameterized by the interval $[a,b]$.*
3. *A curve $\gamma\colon [a,b] \longrightarrow \Omega$ is called closed if $\gamma(a) = \gamma(b)$.*
4. *We say that the curve $\gamma\colon [a,b] \longrightarrow \Omega$ is piecewise continuously differentiable path if there exists a partition $\sigma = \{t_0 = a < t_1 < \ldots < t_n = b\}$ of the interval $[a,b]$ such that the restriction of γ on each interval $]t_j, t_{j+1}[$ can be extended to a continuously differentiable function on $[t_j, t_{j+1}]$*

Remarks

1. *Since for any closed interval* $[a, b]$ *there exists a bijective and differentiable function* $h\colon [0, 1] \longrightarrow [a, b]$*, we can always assume that a given path* γ *is parameterized by* $[0, 1]$*. (We can take* $h(t) = a + (b - a)t$*).*
2. *Let* γ_1 *and* γ_2 *be the two curves defined on the interval* $[0, 2\pi] \longrightarrow \mathbb{C}$ *by* $\gamma_1(t) = \mathrm{e}^{\mathrm{i}t}$ *and* $\gamma_2(t) = \mathrm{e}^{2\mathrm{i}t}$*, these two curves are different but they have by the same range. But nevertheless for the convenience of lecture, if the curve is simple, we identify the curve and its image.*

Definition (Opposite of a Curve)

Let $\gamma\colon [a,b] \longrightarrow \mathbb{C}$ *be a curve. The opposite curve (or the reverse curve) of* γ *denoted by* γ^- *defined by* $\gamma^-(t) = \gamma(a+b-t)$*, for all* $t \in [a,b]$*.* γ *and* γ^- *have the same range or trajectory but they are traveled in the opposite sense.*

Definition (Juxtaposition (or concatenation) of Curves)

Let $\gamma_1 : [a, b] \longrightarrow \mathbb{C}$ and $\gamma_2 : [c, d] \longrightarrow \mathbb{C}$ be two curves such that $\gamma_1(b) = \gamma_2(c)$. The juxtaposition (or the concatenation) of the curves γ_1 and γ_2 is the curve γ denoted by $\gamma_1 \vee \gamma_2$ defined on the interval $[a, b + d - c]$ by

$$\begin{cases} \gamma(t) = \gamma_1(t) & \text{if} \quad t \in [a, b] \\ \gamma(t) = \gamma_2(t - b + c) & \text{if} \quad t \in [b, b + d - c] \end{cases}$$

Remark

$\gamma_1(a)$ is the origin of the curve $\gamma_1 \vee \gamma_2$ and $\gamma_2(d)$ is the end. In particular if $\gamma_1(a) = \gamma_2(d)$, the curve γ is closed, this is the case if $\gamma_1 = \gamma_2^-$

Definition (Equivalent Curves)

Let I_1 and I_2 be two closed and bounded intervals of $\mathbb{R}$.
Two curves $\gamma_1 \colon I_1 \longrightarrow \mathbb{C}$ and $\gamma_2 \colon I_2 \longrightarrow \mathbb{C}$ are called equivalents if there exists an increasing bijective and piecewise continuously differentiable function $h \colon : I_1 \longrightarrow I_2$ such that $\gamma_1 = \gamma_2 \circ h$.

Remark

Any curve $\gamma \colon [a, b] \longrightarrow \mathbb{C}$ is equivalent to a curve on $[0, 1]$. It suffices to take the function $h(t) = \dfrac{t-a}{b-a}$.

Definition (Integration on a curve)

Let f be a continuous function defined on the image of a piecewise continuously differentiable curve γ. The integral of f on the curve γ denoted by $\int_\gamma f(z)dz$ is defined by

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt. \tag{1}$$

Proposition

Let γ_1 and γ_2 be two piecewise continuously differentiable paths on $[a,b]$ and $[c,d]$ respectively. If γ_1 and γ_2 are equivalents, then $\displaystyle\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$, for any continuous function on he image of γ_1.

Proof

Let $h : [a,b] \longrightarrow [c,d]$ be a bijective piecewise continuously differentiable function such that $\gamma_2 \circ h = \gamma_1$.

$$\begin{aligned}\int_c^d f\circ\gamma_2(s)\gamma_2'(s)ds &= \int_a^b f\circ\gamma_2\circ h(t)\gamma_2'(h(t))h'(t)dt \\ &= \int_a^b f\circ\gamma_1(s)\gamma_1'(s)ds.\end{aligned}$$

Proposition

Let γ be a piecewise continuously differentiable path defined on the interval $[a, b]$ and γ^- its opposite path, then

$$\int_{\gamma^-} f(z)dz = -\int_\gamma f(z)dz.$$

Proof

$$\int_{\gamma^-} f(z)dz = -\int_a^b f(\gamma(a+b-t))\gamma'(a+b-t)dt =$$

$$-\int_a^b f(\gamma(s))\gamma'(s)ds = -\int_\gamma f(z)dz. \qquad \square$$

Proposition

If γ_1 and γ_2 are two piecewise continuously differentiable paths juxtaposed, then

$$\int_{\gamma_1 \vee \gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz.$$

Corollary

If γ is a closed piecewise continuously differentiable path, then the integral $\displaystyle\int_{\gamma} f(z)dz$ does not depend on the origin.

Definition

Let $\gamma\colon [a, b] \longrightarrow \mathbb{C}$ be a piecewise continuously differentiable path. The length of γ denoted by $L(\gamma)$ is defined by

$$L(\gamma) = \int_a^b |\gamma'(t)|dt.$$

Consequence

It follows by definition of the integral of function on a curve that for any continuous function f on the range of γ,

$$|\int_\gamma f(z)dz| \leq ML(\gamma), \tag{2}$$

with $M = \sup_{z\in\ \mathrm{range}(\gamma)} |f(z)| = \sup_{t\in[a,b]} |f\circ\gamma(t)|.$

Example

Let $a \in \mathbb{C}$ *and* $r > 0$. *The closed curve* $\gamma\colon [0, 2\pi] \longrightarrow \mathbb{C}$ *defined by* $\gamma(\theta) = a + r\mathrm{e}^{\mathrm{i}\theta}$ *is called the circle of radius* r, *centered at* a *and oriented counterclockwise.*

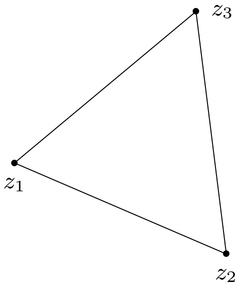
$$\int_\gamma f(z)dz = \mathrm{i}r \int_0^{2\pi} f(a + r\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}\theta} d\theta.$$

If $f(z) = z^n$, $\displaystyle\int_\gamma f(z)dz = 0$ *for every* $n \in \mathbb{Z} \setminus \{-1\}$.

Example

Let z_1 *and* z_2 *be two complex numbers, the curve* γ *defined by* $\gamma(t) = tz_2 + (1-t)z_1,\ t \in [0,1]$ *is called the interval* $[z_1, z_2]$*.*
If $f(z) = z^n$*,* $\displaystyle\int_{[z_1,z_2]} f(z)dz = \frac{1}{n+1}(z_2^{n+1} - z_1^{n+1})$ *for all* $n \in \mathbb{Z} \setminus \{-1\}$*. (In the case* $n \le -1$*, we assume that* $0 \notin \Im\Gamma$*)*

Let z_1, z_2 and z_3 be three different complex numbers.
Let Δ be the triangle of vertices z_1, z_2 and z_3. We denote $\partial\Delta$ the boundary of Δ oriented counterclockwise. We define the closed curve γ by the juxtaposition of the intervals oriented $[z_1, z_2], [z_2, z_3]$ and $[z_3, z_1]$.



This is the curve $\gamma : [0,3] \longrightarrow \mathbb{C}$ defined by

$$\begin{cases} \gamma(t) & = & tz_2 + (1-t)z_1 & 0 \leq t \leq 1 \\ \gamma(t) & = & (t-1)z_3 + (2-t)z_2 & 1 \leq t \leq 2 \\ \gamma(t) & = & (t-2)z_1 + (3-t)z_3 & 2 \leq t \leq 3 \end{cases}$$

We denote γ by $\partial\Delta$.

$$\int_{\partial\Delta} f(z)dz = \int_{[z_1,z_2]} f(z)dz + \int_{[z_2,z_3]} f(z)dz + \int_{[z_3,z_1]} f(z)dz.$$

We remark that if $f(z) = z^n$, then $\displaystyle\int_{\partial\Delta} f(z)dz = 0$, for every $n \in \mathbb{N}$ and also $\displaystyle\int_{\partial\Delta} f(z)dz = 0$, for every $n \in \mathbb{Z}$, $n \leq -2$ and $0 \notin \partial\Delta$.

Consequence

If f is analytic on a neighborhood of Δ, then $\displaystyle\int_{\partial\Delta} f(z)dz = 0$.
(We say that the triangle Δ is in the open subset Ω if its boundary and its interior are in Ω.)

Theorem (Index of a closed curve)

Let γ be a closed piecewise continuously differentiable path in $\mathbb{C}$ and Ω the complement of the range of γ in $\mathbb{C}$. For every $z \in \Omega$ we define the index or the winding number of γ at z by

$$I(\gamma, z) = \frac{1}{2\mathrm{i}\pi} \int_a^b \frac{\gamma'(t)}{\gamma(t) - z} dt. \tag{3}$$

$I(\gamma, z)$ is a function with values in $\mathbb{Z}$, constant on each connected component of Ω and equal to zero on the unbounded connected component of Ω.

Proof

If $\gamma\colon [a,b] \longrightarrow \mathbb{C}$ and $\sigma = \{a_0 = a, \ldots, a_n = b\}$ the partition associated to σ, (i.e. $\gamma_j = \gamma_{\restriction]a_{j-1}, a_j[}$ is differentiable), then

$$\mathrm{Ind}(\gamma, z) = \frac{1}{2\mathrm{i}\pi} \int_a^b \frac{\gamma'(t)}{\gamma(t) - z} dt.$$

We define the function ψ on the interval $[a,b]$ by

$$\psi(s) = \exp \int_a^s \frac{\gamma'(t)}{\gamma(t) - z} dt.$$

To prove that $\mathrm{Ind}(\gamma, z) \in \mathbb{Z}$, it suffices to show that $\psi(b) = 1$ because $\exp w = 1 \Leftrightarrow w \in 2\mathrm{i}\pi\mathbb{Z}$.
For every s such that $\gamma'(s)$ is defined, we have

$$\frac{\psi'(s)}{\psi(s)} = \frac{\gamma'(s)}{\gamma(s) - z} \Leftrightarrow \psi'(s)(\gamma(s) - z) = \psi(s)\gamma'(s).$$

It follows that the derivative of $\dfrac{\psi(s)}{\gamma(s)-z}$ is zero on each interval $]a_j, a_{j+1}[$ and then it is a constant on each interval $[a_j, a_{j+1}]$ on which γ is continuously differentiable, thus it is constant on $[a, b]$. It follows that $\dfrac{\psi(a)}{\gamma(a)-z} = \dfrac{\psi(b)}{\gamma(b)-z}$ and as $\gamma(a) = \gamma(b)$, we deduce that $\psi(b) = \psi(a) = 1$, $(\psi(a) = e^0 = 1)$.
$\dfrac{\gamma'(t)}{\gamma(t)-z}$ is a continuous function on $[a, b]$ which is deduced from theorem of continuity of integral which depends on a parameter. Furthermore, the function $z \longmapsto \mathrm{Ind}(\gamma, z)$ is continuous on Ω, with entire values, it is then constant on each connected component of Ω.

Furthermore, $\lim_{|z|\to+\infty} \mathrm{Ind}(\gamma, z) = 0$, thus the map $z \longmapsto \mathrm{Ind}(\gamma, z)$ is zero on the unbounded connected component of Ω. $\square$

Remark

If γ is a closed piecewise continuously differentiable path. The index of γ at a point $z \notin \mathrm{range}(\gamma)$ is intuitively equal to the number of turns described by γ around z when t describes the interval $[a, b]$.

Proposition

If γ is the circle of radius r and centered at a with positive orientation, then $\mathrm{Ind}(\gamma, z) = 1$ if $|z-a| < r$ and $\mathrm{Ind}(\gamma, z) = 0$ if $|z-a| > r$.

Proof

From the previous theorem, it suffices to find $\mathrm{Ind}(\gamma, a)$

$$\mathrm{Ind}(\gamma, a) = \frac{1}{2\mathrm{i}\pi}\int_\gamma \frac{dz}{z-a} = \frac{1}{2\mathrm{i}\pi}\int_0^{2\pi} \frac{\mathrm{i}re^{\mathrm{i}\theta}}{re^{\mathrm{i}\theta}}d\theta = 1.$$

$\square$

Theorem

Let $f:\ \Omega \longrightarrow \mathbb{C}$ be a continuous function on an open subset Ω of $\mathbb{C}$. The function f has a primitive on Ω, if and only if, for any closed piecewise continuously differentiable path γ in Ω $\displaystyle\int_\gamma f(z)dz = 0.$

Proof

Let F be a primitive of f (i.e. $F' = f$ on Ω) and let $\gamma\colon [a,b] \longrightarrow \Omega$ be a closed piecewise continuously differentiable path, we have

$$\int_\gamma f(z)dz = \int_a^b F'(\gamma(t))\gamma'(t)dt = F(\gamma(b)) - F(\gamma(a)) = 0,$$

(because $\gamma(a) = \gamma(b)$).

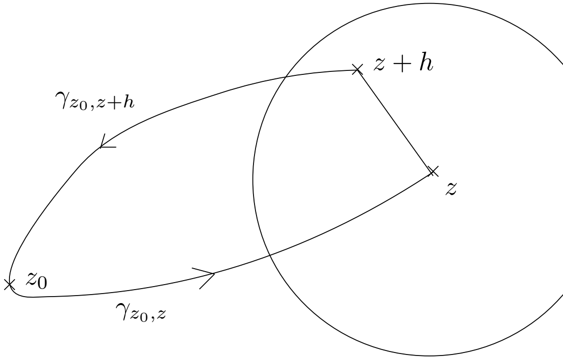
Conversely: Assume that $\int_\gamma f(z)dz = 0$ for any closed piecewise continuously differentiable path γ. It suffices to construct a primitive on each connected component of Ω. We can then assume that Ω is connected, then any two points in Ω can be joined by a piecewise continuously differentiable path in Ω. Let $z_0 \in \Omega$, for every $z \in \Omega$, there exists a piecewise continuously differentiable path of origin z_0 and of end z. Let γ_1 and γ_2 be two such curves, one has

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz = 0,$$

with γ the closed curve obtained by juxtaposition of the curve γ_1 and of the opposite curve of γ_2. We set F the function defined on Ω by

$$F(z) = \int_{\gamma_{z_0,z}} f(w)dw,$$

with $\gamma_{z_0,z}$ be a piecewise continuously differentiable path of origin z_0 and of end z. The function F is well defined and it F is independent of the choice of $\gamma_{z_0,z}$.
Let us prove that F is a primitive of f.


$\gamma_{z_0,z+h}$
$z+h$
z
z_0
$\gamma_{z_0,z}$

Let $r > 0$ be such that $D(z, r) \subset \Omega$ and let h be small enough such that $|h| < r$. We consider the integral of f on the closed curve γ obtained by juxtaposing the curves $\gamma_{z_0,z}$, $[z, z + h]$ and the opposite curve of any $\gamma_{z_0,z+h}$.

$$\int_\gamma f(w)dw = \int_{\gamma_{z_0,z}} f(w)dw + \int_{[z,z+h]} f(w)dw - \int_{\gamma_{z_0,z+h}} f(w)dw = 0.$$

Thus

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h}\int_{[z,z+h]} (f(w) - f(z))dw.$$

It follows that

$$\lim_{h\to 0} \left|\frac{F(z+h) - F(z)}{h} - f(z)\right| \leq \lim_{h\to 0} \sup_{w\in[z,z+h]} |f(w) - f(z)| = 0.$$

Thus F is differentiable and $F'(z) = f(z)$.

Corollary

Since $\frac{z^{n+1}}{n+1}$ *is a primitive of* z^n, *for* $n \in \mathbb{Z} \setminus \{-1\}$, *then* $\int_\gamma z^n dz = 0$ *for any closed piecewise continuously differentiable path* γ, *whenever* $n \in \mathbb{Z} \setminus \{-1\}$, *with* $0 \notin \operatorname{range}(\gamma)$ *if* $n \leq -2$.

Definition

• *A set* $E \subset \mathbb{C}$ *is convex if for each pair of points* $a, b \in E$, *we have* $[a, b] \subset E$.
• E *is starlike with respect to* $a \in E$ *(*a *called a star center of* E*) if* $[a, z] \subset E$ *for each* $z \in E$. *Note that any non-empty convex set is starlike with respect any of its points and the starlike sets are polygonally connected.*

Theorem

(Cauchy 1825)
Let f be a continuous function on a **convex** *open set Ω such that*

$$\int_{\partial\Delta} f(z)dz = 0,$$

for any triangle Δ in Ω, then f has a primitive on Ω.

(The result is still true if Ω is starlike).

Proof

Let $z_0 \in \Omega$ be fixed and $z \in \Omega$, the interval $[z_0, z] \subset \Omega$. We set

$$F(z) = \int_{[z_0,z]} f(w)dw.$$

If $\lambda \in \mathbb{C}^*$ is such that $z + \lambda \in \Omega$, the integral of f on the boundary of the triangle $\Delta(z_0, z, z + \lambda)$ is zero thus we have

$$\frac{F(z+\lambda) - F(z)}{\lambda} - f(z) = \frac{1}{\lambda}\int_{[z,z+\lambda]} f(w)dw - \frac{1}{\lambda}\int_{[z,z+\lambda]} f(z)dw.$$

The proof can be completed as in theorem 2.1, thus $F'(z) = f(z)$.
$\square$

Lemma

(Topology Lemma)
Let $(K_n)_n$ be a decreasing sequence of non-empty compacts sets of $\mathbb{C}$ such that the sequence $(\delta(K_n))_n$ tends to 0 when n tends to $+\infty$, ($\delta(K_n)$ is the diameter of K_n). Then $\bigcap_n K_n$ is reduced to a point.

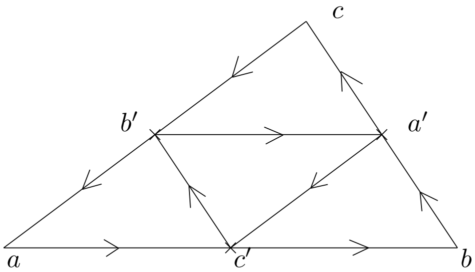
Theorem

(Cauchy's Theorem for a triangle)
Let Ω be an open subset of $\mathbb{C}$ and $f:\ \Omega \longrightarrow \mathbb{C}$ be a holomorphic function on Ω. Then for any triangle Δ in Ω, $\displaystyle\int_{\partial\Delta} f(w)dw = 0$.

Proof

Let $\Delta(a, b, c)$ be an oriented triangle Δ and a', b', c' the middle points of the intervals $[b, c], [c, a]$ and respectively $[a, b]$. We consider the four triangles Δ^j $(1 \leq j \leq 4)$ positively oriented, respectively (a, c', b'), (b, a', c'), (c, b', a') and (a', b', c'). We set

$$J = \int_{\partial\Delta} f(z)dz = \sum_{j=1}^{4} \int_{\partial\Delta^j} f(z)dz.$$


c
b′
a′
a
c′
b

There exist $1 \leq j \leq 4$ such that $|\int_{\partial\Delta^j} f(z)dz| \geq \frac{|J|}{4}$. We denote this triangle by Δ_1. We apply to Δ_1 the same method and we construct a triangle $\Delta_2 \subset \Delta_1$ such that

$$|\int_{\partial\Delta_2} f(z)dz| \geq \frac{1}{4}|\int_{\partial\Delta_1} f(z)dz| \geq \frac{|J|}{4^2}.$$

By induction, we construct a decreasing sequence of triangles $(\Delta_n)_{n\in\mathbb{N}}$ such that $\forall n \in \mathbb{N}$,

$$|\int_{\partial\Delta_n} f(z)dz| \geq \frac{1}{4}|\int_{\partial\Delta_{n-1}} f(z)dz| \geq \frac{|J|}{4^n} \tag{4}$$

By construction $L(\partial\Delta_n) = \frac{1}{2^n} L(\partial\Delta)$. ($L(\partial\Delta)$ is the length of $\partial\Delta$).
We denote by $\{z_0\} = \bigcap_n \Delta_n$. Since f is differentiable at z_0, then $\forall\ \varepsilon > 0,\ \exists\ r > 0$ such that

$$|f(z) - f(z_0) - (z - z_0)f'(z_0)| \leq \varepsilon|z - z_0| \quad for |z - z_0| \leq r.$$

Since $\delta(\Delta_n) \underset{n\to+\infty}{\longrightarrow} 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and for all $z \in \Delta_n$, $|z - z_0| \leq r$. We deduce that

$$\begin{aligned} \left|\int_{\partial\Delta_n} f(z)dz\right| &= \left|\int_{\partial\Delta_n} (f(z)-f(z_0)-(z-z_0)f'(z_0))dz\right| \le \varepsilon \sup_{z\in\Delta_n} |z - \\ &\le \varepsilon(L(\partial\Delta_n))^2 \le \varepsilon\left(\frac{L(\partial\Delta)}{2^n}\right)^2. \end{aligned}$$

We deduce by inequality (4) that $\forall n \in \mathbb{N}$, $\dfrac{|J|}{4^n} \le \dfrac{\varepsilon}{4^n}(L(\partial\Delta))^2$, which yields that $|J| \le \varepsilon(L(\partial\Delta))^2$, $\forall \varepsilon > 0$, thus $J = 0$. $\square$

Remark

Theorem 3.2 remains valid for a rectangle. It suffices to divide the rectangle to 4 isometrics rectangles and repeat the same proof as before.

Theorem

Let $f: \ \Omega \longrightarrow \mathbb{C}$ be a continuous function and $z_0 \in \Omega$. Assume that f is holomorphic on $\Omega \setminus \{z_0\}$. Then for any triangle $\Delta \subset \Omega$, $\displaystyle\int_{\partial\Delta} f(z)dz = 0$.

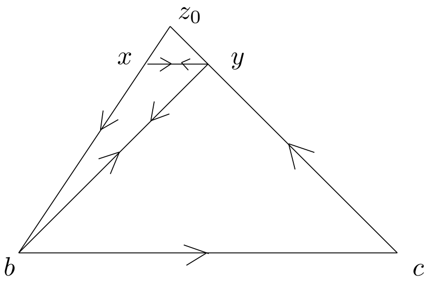
Proof

Let $\Delta(a, b, c)$ be a triangle in Ω.

• If $z_0 \notin \overline{\Delta}$, then f is holomorphic on $\Omega \setminus \{z_0\}$ and $\Delta \subset \Omega \setminus \{z_0\}$.

Then $\displaystyle\int_{\partial\Delta} f(z)dz = 0$.

If $z_0 = a$ is a vertex of Δ.


z_0
x
y
b
c

Let x and y be two points situated respectively on $]a, b]$ and $]a, c]$ and close to a.

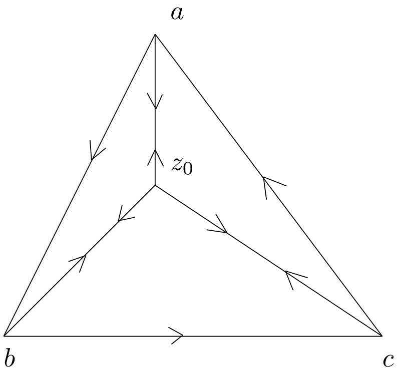
$$\int_{\partial\Delta} f(z)dz = \int_{[a,x,y]} f(z)dz + \int_{[x,b,y]} f(z)dz + \int_{[y,b,c]} f(z)dz.$$

Then from the previous case $\displaystyle\int_{[x,b,y]} f(z)dz = \int_{[y,b,c]} f(z)dz = 0$, thus

$$\left|\int_{\partial\Delta} f(z)dz\right| = \left|\int_{[a,x,y]} f(z)dz\right| \leq \sup_{z\in\Delta} |f(z)| L([a, x, y]),$$

x and y being arbitrary, it follows that $L([a, x, y]) \underset{x,y\longrightarrow a}{\longrightarrow} 0$, thus $\displaystyle\int_{\partial\Delta} f(z)dz = 0.$

• If z_0 is in Δ and it is not a vertex, then we take the 3 triangles $[z_0, a, b]$, $[z_0, b, c]$ and $[z_0, c, a]$, and the result is deduced from the previous case.



Theorem

[Cauchy's Theorem (1831) on convex domains]
Let Ω *be a convex open subset of* $\mathbb{C}$ *and* $f : \Omega \longrightarrow \mathbb{C}$ *a continuous function and holomorphic on* $\Omega \setminus \{z_0\}$, $(z_0 \in \Omega)$. *Then* f *has a primitive on* Ω *and for any closed piecewise continuously differentiable path* γ *in* Ω, $\displaystyle\int_\gamma f(z)dz = 0$.

Proof

The theorem results from theorems 2.1, 2.4 and 3.2. □

Theorem

Let Ω be an open subset of $\mathbb{C}$ which contains the closed disc $\overline{D(z_0, r)}$, $r > 0$ and let $f : \ \Omega \longrightarrow \mathbb{C}$ be a holomorphic function. Then for all $z \in D(z_0, r)$

$$f(z) = \frac{1}{2\mathrm{i}\pi} \int_\gamma \frac{f(w)}{w - z} dw, \tag{5}$$

with $\gamma \colon [0, 2\pi] \longrightarrow \mathbb{C}$ the closed curve defined by $\gamma(t) = z_0 + r\mathrm{e}^{\mathrm{i}t}$.

Proof

There exists $\varepsilon > 0$ such that $D(z_0, r+\varepsilon) \subset \Omega$. Let g be the function defined on Ω by

$$g(w) = \begin{cases} f'(z) & \text{if} \quad w = z \\ \dfrac{f(w)-f(z)}{w-z} & \text{if} \quad w \neq z \end{cases}.$$

g is holomorphic on $\Omega \setminus \{z\}$ and continuous on Ω. By theorem 3.5 (the convex set is $D(z_0, r+\varepsilon)$).

$$\int_\gamma g(w)dw = 0 = \int_\gamma \frac{f(w)}{w-z}dw - \int_\gamma \frac{f(z)}{w-z}dw.$$

Thus $f(z) = \dfrac{1}{2i\pi}\displaystyle\int_\gamma \frac{f(w)}{w-z}dw$ (because $\displaystyle\int_\gamma \frac{dw}{w-z} = 2i\pi$). □

Theorem

A function $f : \Omega \longrightarrow \mathbb{C}$ is holomorphic on Ω if and only if f is analytic on Ω and $\forall\ z_0 \in \Omega$,

$$f(z) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k \tag{6}$$

This series converges on any disc centered at z_0 in Ω.

Proof

Let $r > 0$ be such that $\overline{D(z_0, r)} \subset \Omega$. We have:
$f(z) = \dfrac{1}{2\mathrm{i}\pi}\displaystyle\int_\gamma \dfrac{f(w)}{w - z}dw$, where $\gamma(\theta) = z_0 + r\mathrm{e}^{\mathrm{i}\theta}$, $\theta \in [0, 2\pi]$ and $|z - z_0| < r$.

$$\frac{1}{w - z} = \frac{1}{(w - z_0)(1 - \frac{z - z_0}{w - z_0})} = \sum_{n=0}^{+\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}}.$$

$$f(z) = \sum_{n=0}^{+\infty}(z - z_0)^n \frac{1}{2\mathrm{i}\pi}\int_\gamma \frac{f(w)}{(w - z_0)^{n+1}}dw = \sum_{n=0}^{+\infty} a_n(z - z_0)^n.$$

The series converges on the disc $D(z_0, r)$. Then

1. $a_n = \dfrac{f^{(n)}(z_0)}{n!} = \dfrac{1}{2\mathrm{i}\pi}\displaystyle\int_\gamma \dfrac{f(w)}{(w-z_0)^{n+1}}dw =$ $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \dfrac{f(z_0+r\mathrm{e}^{\mathrm{i}\theta})}{(r\mathrm{e}^{\mathrm{i}\theta})^n}d\theta.$
2. The series $\sum_{n\geq 0} a_n(z-z_0)^n$ converges on any disc centered at z_0 in Ω.
   The converse is given by theorem **??** chapter I.

□

Corollary

Let Ω be an open subset of $\mathbb{C}$. If f is holomorphic on Ω, then for all $n \in \mathbb{N}$, $f^{(n)}$ is holomorphic on Ω.

Theorem

Let Ω be a domain of $\mathbb{C}$ and $f : \Omega \longrightarrow \mathbb{C}$ be a holomorphic function. If f is not identically zero, then the zeros of f are isolated.

Theorem

The ring of holomorphic functions on a domain Ω of $\mathbb{C}$ is integral.

These two theorems are deduced from the fact that any holomorphic function is analytic.

Theorem (Morera's Theorem)

Let f be a continuous function on Ω. Then f is holomorphic on Ω if and only if for any triangle $\Delta \subset \Omega$ (interior included), the integral $\displaystyle\int_{\partial\Delta} f(w)dw = 0$.

Proof

The necessary condition is given by theorem 3.2. For the sufficient condition, the hypothesis that $\displaystyle\int_{\partial\Delta} f(w)dw = 0$ for any triangle in Ω yields that, locally the function f has a primitive F. F is holomorphic, then f is also holomorphic. $\square$

Corollary

If $f \in \mathcal{H}(\Omega \setminus \{z_0\})$ and continuous on Ω, then $f \in \mathcal{H}(\Omega)$. ($z_0 \in \Omega$).

Proof

Theorem 3.4 yields that $\displaystyle\int_{\partial\Delta} f(w)dw = 0$ for any triangle Δ in Ω.
Then by Morera's theorem, f is holomorphic on Ω. $\square$

Corollary

Let f be a continuous function on an open set Ω. The following conditions are equivalents.
i) f is holomorphic on Ω,
ii) f is analytic on Ω,
iii) locally f has a primitive on Ω,
iv) For any triangle $\Delta \subset \Omega$, $\displaystyle\int_{\partial\Delta} f(z)dz = 0$.

Proof

i) $\Leftrightarrow$ ii) results from theorem 4.2.

iii) $\Rightarrow$ i) Locally f has a primitive F, F is holomorphic, thus F' is holomorphic by Corollary 4.3.

i) $\Rightarrow$ iv) this is theorem 3.5.

iv) $\Rightarrow$ iii) Results from theorem 2.4. $\square$