Chapter 24

# Gauss's Law

**Karl Friedrich Gauss**
German mathematician and astronomer (1777–1855)

## CHAPTER OUTLINE

# 24.1 Electric Flux

*Latin: flux = "to flow"*

*Graphically*: Electric flux $\Phi_E$ represents the number of E-field lines crossing a surface.

The number of lines per unit area (the *line density*) is proportional to the magnitude of the electric field.

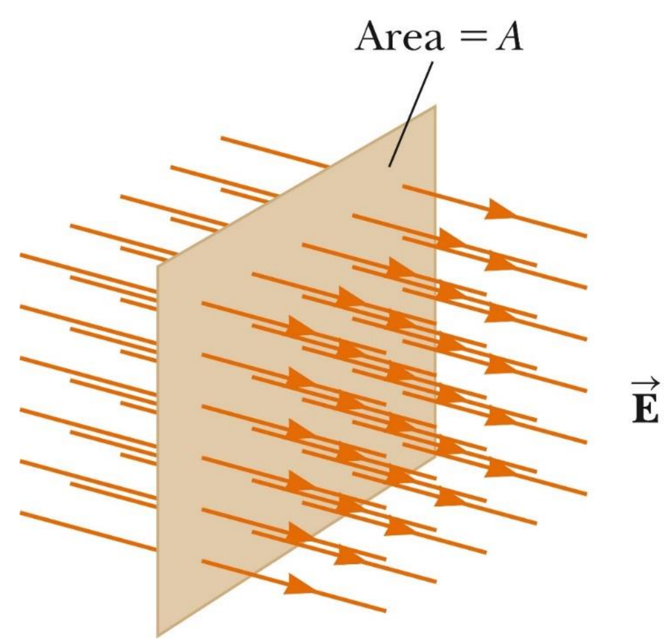
***Electric flux***

is the product of the magnitude of the electric field E and the surface area, *A*, perpendicular to the field.

$$\Phi_E = EA$$

**Electric flux $\Phi_E$ is proportional to the number of electric field lines penetrating some surface.**

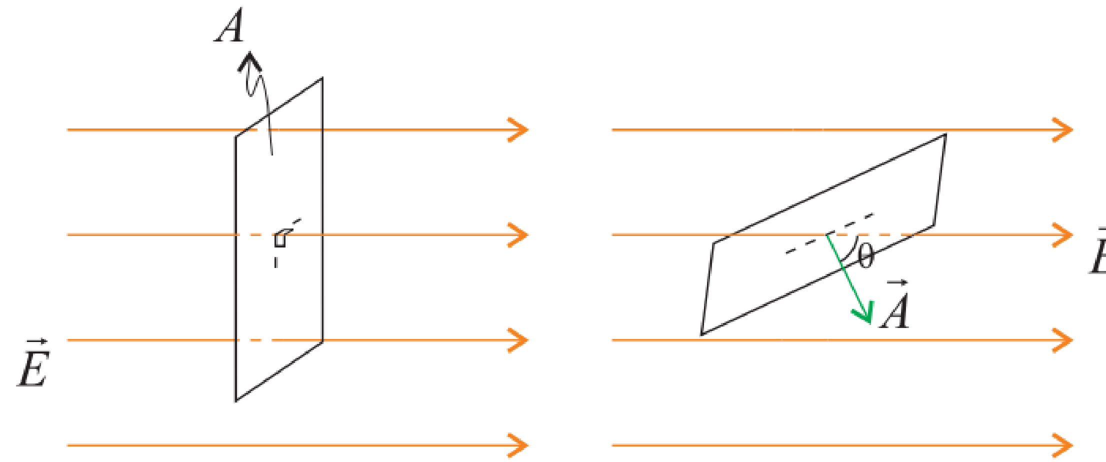
*Units: $N \cdot m^2 / C$*




Field lines representing a uniform electric field penetrating a plane of area *A* perpendicular to the field. The electric flux $\Phi_E$ through this area is equal to *EA*.

# 24.1 Electric Flux

*Mathematically*:




$$\Phi_E = EA \qquad \Phi_E = \vec{E}\cdot\vec{A}$$

Reminder: Vector of the area $\vec{A}$ is perpendicular to the area A.

**Example 24.1 Electric Flux Through a Sphere**

What is the electric flux through a sphere that has a radius of 1.00 m and carries a charge of $+1.00\ \mu C$ at its center?

***Solution*** The magnitude of the electric field 1.00 m from this charge is found using Equation 23.9:

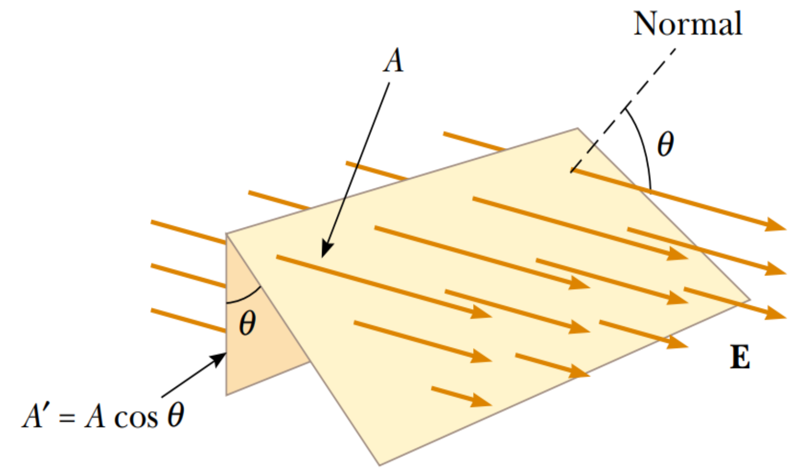
$$E = k_e \frac{q}{r^2} = (8.99 \times 10^9\ \mathrm{N\cdot m^2/C^2})\frac{1.00\times10^{-6}\ \mathrm{C}}{(1.00\ \mathrm{m})^2}$$
$$= 8.99 \times 10^3\ \mathrm{N/C}$$

The field points radially outward and is therefore everywhere perpendicular to the surface of the sphere. The flux through the sphere (whose surface area $A = 4\pi r^2 = 12.6\ \mathrm{m^2}$) is thus

$$\Phi_E = EA = (8.99\times10^3\ \mathrm{N/C})(12.6\ \mathrm{m^2})$$
$$= \boxed{1.13\times10^5\ \mathrm{N\cdot m^2/C}}$$

# 24.1 Electric Flux

**If the surface is not perpendicular to the field**




The normal to the surface of area $A$ is at an angle $\theta$ to the uniform electric field.

$$\Phi_E = EA\cos\theta$$

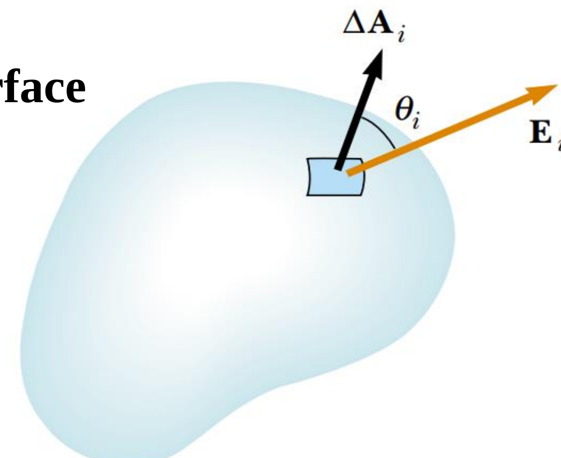
- Flux through a surface of fixed area $A$ has a *maximum* value $EA$ when the surface is perpendicular to the field (when the normal to the surface is parallel to the field, that is, $\theta = 0°$

- Flux is *zero* when the surface is parallel to the field (when the normal to the surface is perpendicular to the field, that is, $\theta = 90°$.

**Quick Quiz 24.1** Suppose the radius of the sphere in Example 24.1 is changed to 0.500 m. What happens to the flux through the sphere and the magnitude of the electric field at the surface of the sphere? (a) The flux and field both increase. (b) The flux and field both decrease. (c) The flux increases and the field decreases. (d) The flux decreases and the field increases. (e) The flux remains the same and the field increases. (f) The flux decreases and the field remains the same.

# 24.1 Electric Flux

**More general situations, the electric field may vary over a surface**




$$\Phi_E = EA' = EA\cos\theta$$

That means, this equation is only valid over a small element of area $\Delta A_i$

A small element of surface area $\Delta A_i$. The electric field makes an angle $\theta_i$ with the vector $\Delta A_i$, defined as being normal to the surface element

The electric flux $\Delta\Phi_E$ through element $i$ is

$$\Delta\Phi_E = E_i \Delta A_i \cos\theta_i = \vec{\mathbf{E}}_i \cdot \Delta\vec{\mathbf{A}}_i$$

$$\Phi_E = \lim_{\Delta A_i \to 0} \sum E_i \cdot \Delta A_i$$

$$\Phi_E = \int_{\text{surface}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

- ❖ The surface integral (integral must be evaluated over the surface)
- ❖ In general, the value of the flux $\Phi_E$ will depend both on the *field pattern* and *on the surface.*

# 24.1 Electric Flux (Closed Surface)

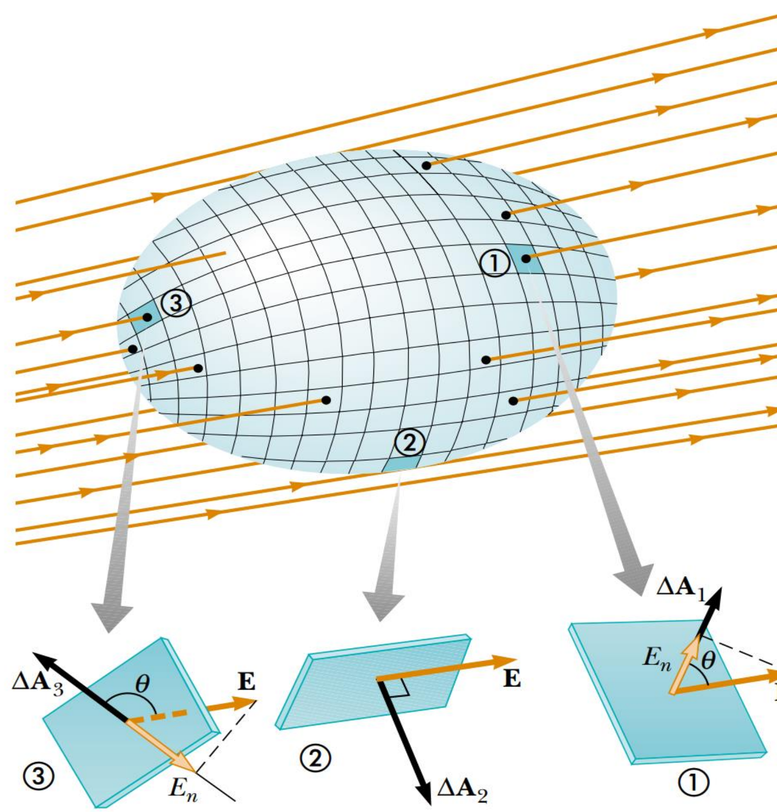
Flux through a *closed surface*, is defined as one that divides space into an inside and an outside region.

A closed surface in an electric field. The area vectors $\Delta A_i$ point in different directions and are normal to the surface and point outward.

Element (1), the field lines are crossing the surface from the inside to the outside and ) $\theta < 90°$ ,
$\Phi$ is positive

Element (2), the field lines are perpendicular to the vector $\Delta A_1$ and ) $\theta = 90°$ ,
$\Phi$ is zero

Element (3), the field lines are crossing the surface from outside to inside, $180° > \theta > 90°$
$\Phi$ is negative because $\cos\theta$ is negative




The flux through an area element can be positive (element 1), zero (element 2), or negative (element 3).

This net flux through surface is the number of lines leaving the surface minus the number entering the surface.

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \oint E_n \, dA$$

## Example 24.2 Flux Through a Cube

Consider a uniform electric field **E** oriented in the $x$ direction. Find the net electric flux through the surface of a cube of edge length $\ell$, oriented as shown in Figure 24.5.

***Solution*** The net flux is the sum of the fluxes through all faces of the cube. First, note that the flux through four of the faces (③, ④, and the unnumbered ones) is zero because **E** is perpendicular to $d\mathbf{A}$ on these faces.

The net flux through faces ① and ② is

$$\Phi_E = \int_1 \mathbf{E}\cdot d\mathbf{A} + \int_2 \mathbf{E}\cdot d\mathbf{A}$$

For face ①, **E** is constant and directed inward but $d\mathbf{A}_1$ is directed outward ($\theta = 180°$); thus, the flux through this face is

$$\int_1 \mathbf{E}\cdot d\mathbf{A} = \int_1 E(\cos 180°)\, dA = -E\int_1 dA = -EA = -E\ell^2$$

because the area of each face is $A = \ell^2$.

For face ②, **E** is constant and outward and in the same direction as $d\mathbf{A}_2$ ($\theta = 0°$); hence, the flux through this face is

$$\int_2 \mathbf{E}\cdot d\mathbf{A} = \int_2 E(\cos 0°)\, dA = E\int_2 dA = +EA = E\ell^2$$

Therefore, the net flux over all six faces is

$$\Phi_E = -E\ell^2 + E\ell^2 + 0 + 0 + 0 + 0 = 0$$

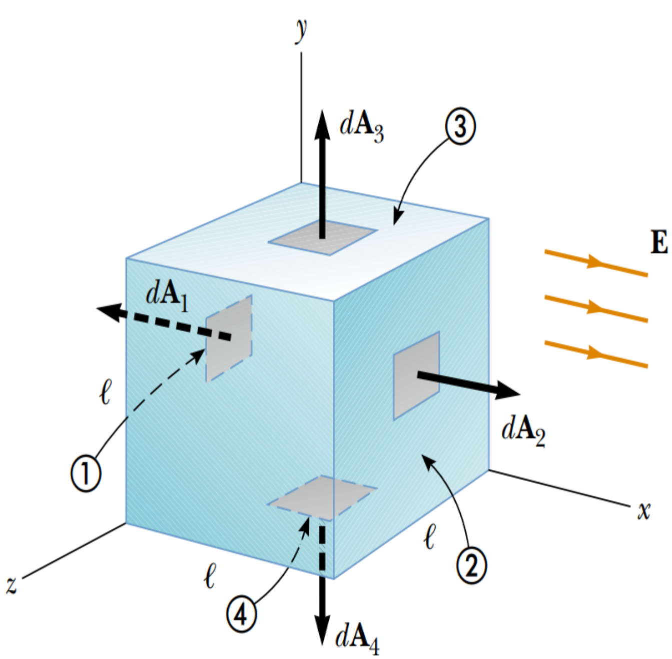



**Figure 24.5** (Example 24.2) A closed surface in the shape of a cube in a uniform electric field oriented parallel to the $x$ axis. Side ④ is the bottom of the cube, and side ① is opposite side ②.

## 24.1 Electric Flux **problem (24.4)**

**4.** Consider a closed triangular box resting within a horizontal electric field of magnitude $E = 7.80 \times 10^4$ N/C as shown in Figure P24.4. Calculate the electric flux through (a) the vertical rectangular surface, (b) the slanted surface, and (c) the entire surface of the box.

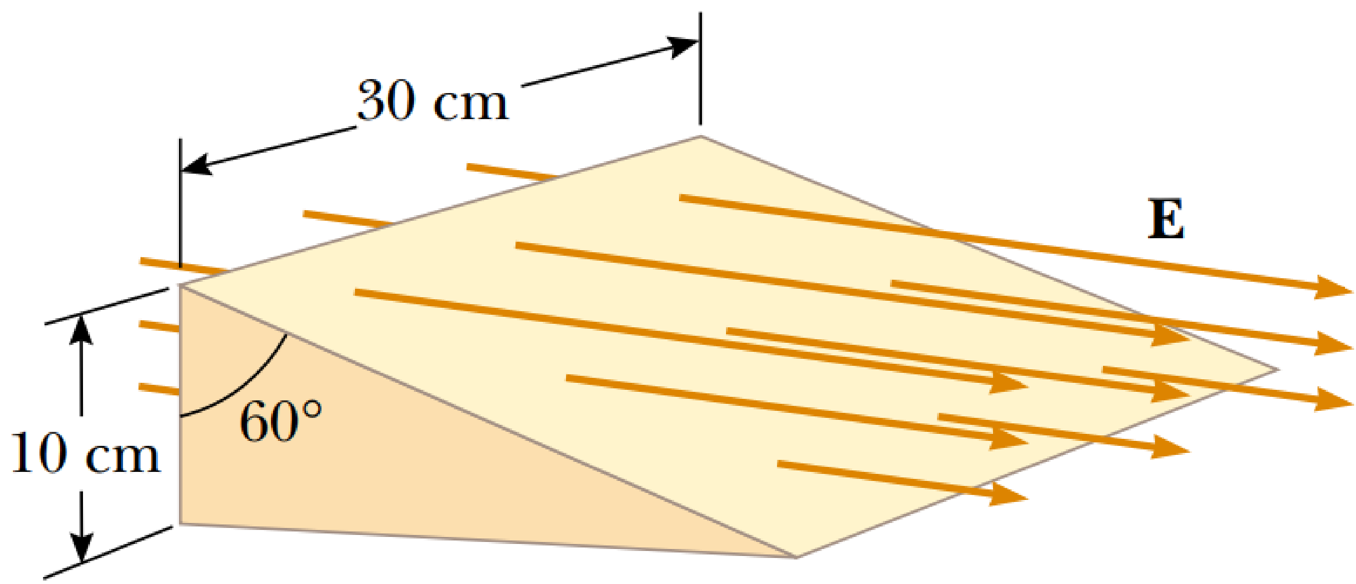
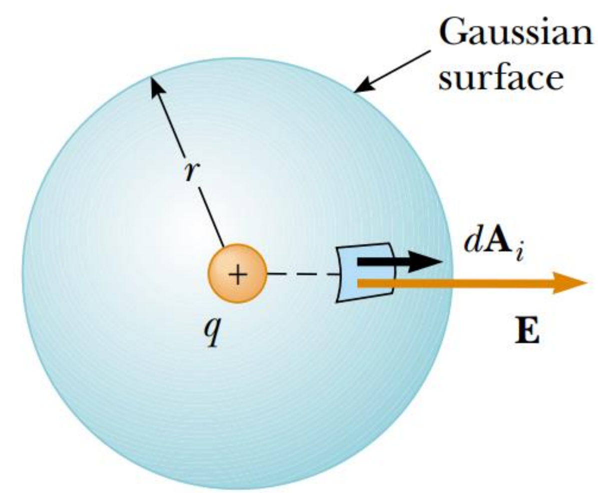



**Figure P24.4**

# 24.2 Gauss's Law

Shows the relationship between the ***net electric flux*** through a closed surface (often called a *gaussian surface*) and the ***charge enclosed*** by the surface.




**A positive point charge *q* case**

**E** is parallel to the vector $\Delta A_1$

$$\mathbf{E} \cdot \Delta \mathbf{A}_i = E \, \Delta A_i$$

A spherical gaussian surface of radius *r* surrounding a point charge *q*. When the charge is at the center of the sphere, the electric field is everywhere normal to the surface and constant in magnitude.

The net flux through the gaussian surface is

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \oint E \, dA = E \oint dA$$

***E*** outside of the integral because, by symmetry, *E* is constant over the surface.

# 24.2 Gauss's Law

The gaussian surface is spherical

$$\oint dA = A = 4\pi r^2.$$

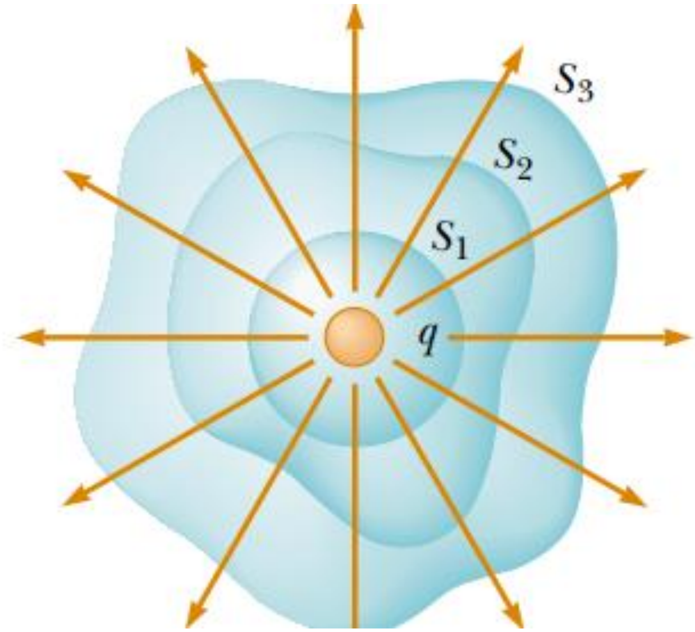
The net flux through the gaussian surface is

$$\Phi_E = \frac{k_e q}{r^2}(4\pi r^2) = 4\pi k_e q$$

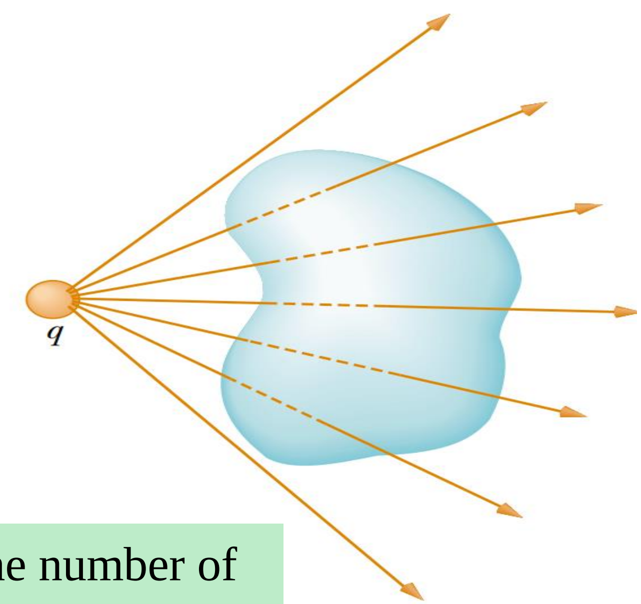
$$k_e = 1/4\pi\epsilon_0,$$

$$\Phi_E = \frac{q}{\epsilon_0}$$

$$\epsilon_0 = (8.85 \times 10^{-12}\ \mathrm{C^2/N\cdot m^2})$$



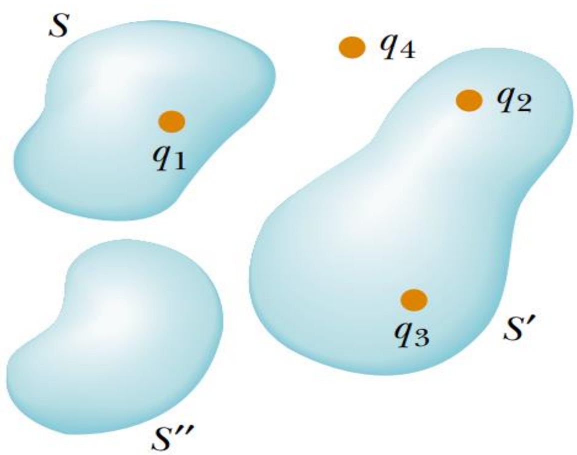

Closed surfaces of various shapes surrounding a charge $q$. The net electric flux is the same through all surfaces.




A point charge located *outside* a closed surface. The number of lines entering the surface equals the number leaving the surface

# 24.2 Gauss's Law

several closed surfaces surrounding a charge $q$




The net electric flux through any closed surface depends only on the charge *inside* that surface. The net flux through surface $S$ is $q_1/\varepsilon_0$, the net flux through surface $S'$ is $(q_2 + q_3)/\varepsilon_0$, and the net flux through surface $S''$ is zero. Charge $q_4$ does not contribute to the flux through any surface because it is outside all surfaces.

**The net flux through *any* closed surface surrounding a point charge *q* is given by $q/\varepsilon_0$ and is independent of the shape of that surface.**

**The net electric flux through a closed surface that surrounds no charge is zero.**

**Gauss's law,** states that the net flux through *any* closed surface is

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{in}}}{\epsilon_0}$$

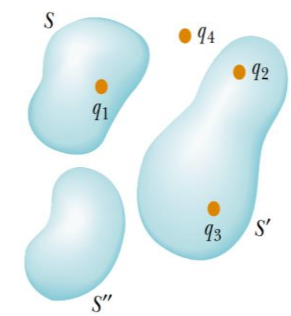
where $q_{\text{in}}$ represents the net charge inside the surface and **E** represents the electric field at any point on the surface.

Gauss's law states that the electric *flux* is proportional to the enclosed charge, not the electric *field*.

# 24.2 Gauss's Law

**Quick Quiz 24.3** If the net flux through a gaussian surface is *zero*, the following four statements *could be true*. Which of the statements *must be true?* (a) There are no charges inside the surface. (b) The net charge inside the surface is zero. (c) The electric field is zero everywhere on the surface. (d) The number of electric field lines entering the surface equals the number leaving the surface.

**Quick Quiz 24.4** Consider the charge distribution shown in Figure 24.9. The charges contributing to the total electric *flux* through surface $S'$ are (a) $q_1$ only (b) $q_4$ only (c) $q_2$ and $q_3$ (d) all four charges (e) none of the charges.




**Conceptual Example 24.3 Flux Due to a Point Charge**

A spherical gaussian surface surrounds a point charge $q$. Describe what happens to the total flux through the surface if

**(A)** the charge is tripled,

**(B)** the radius of the sphere is doubled,

**(C)** the surface is changed to a cube, and

**(D)** the charge is moved to another location inside the surface.

***Solution***

(A) The flux through the surface is tripled because flux is proportional to the amount of charge inside the surface.

(B) The flux does not change because all electric field lines from the charge pass through the sphere, regardless of its radius.

(C) The flux does not change when the shape of the gaussian surface changes because all electric field lines from the charge pass through the surface, regardless of its shape.

(D) The flux does not change when the charge is moved to another location inside that surface because Gauss's law refers to the total charge enclosed, regardless of where the charge is located inside the surface.

# 24.2 Gauss's Law

**11.** Four closed surfaces, $S_1$ through $S_4$, together with the charges $-2Q$, $Q$, and $-Q$ are sketched in Figure P24.11. (The colored lines are the intersections of the surfaces with the page.) Find the electric flux through each surface.

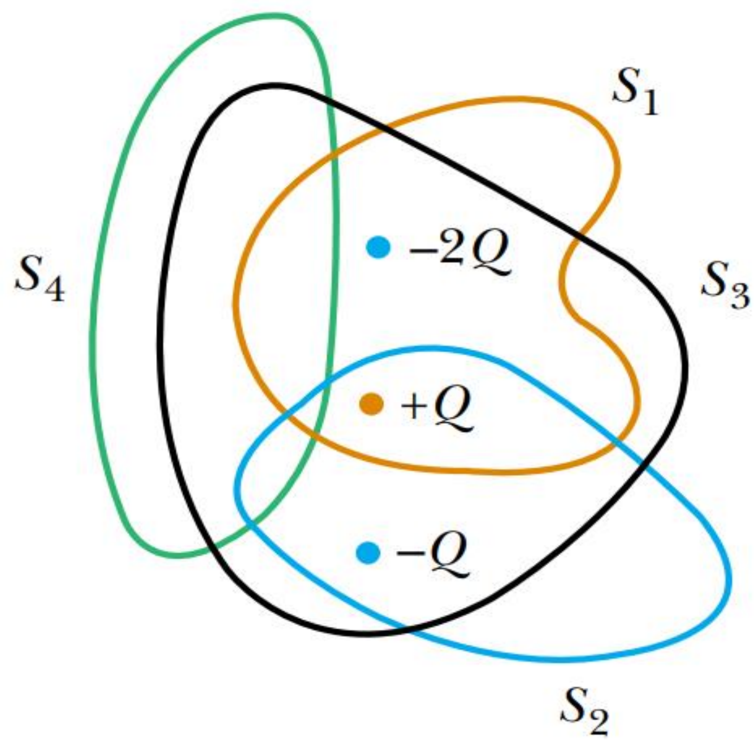


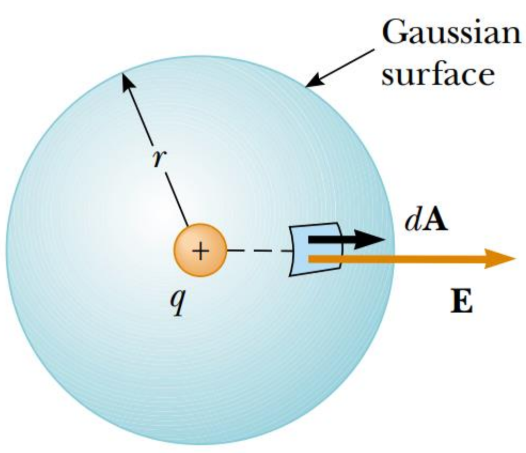

**Figure P24.11**

# 24.3 Application of Gauss's Law to Various Charge Distributions

**Example 24.4 The Electric Field Due to a Point Charge**

Starting with Gauss's law, calculate the electric field due to an isolated point charge ***q***

**E** is // to $d$**A** at each point. Therefore, **E.**$d$**A** = $E\ dA$




***Gauss's law gives***

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \oint E\, dA = \frac{q}{\epsilon_0}$$

By symmetry, $E$ is constant everywhere on the surface, which satisfies condition (1), so it can be removed from the integral. Therefore,

$$\oint E\, dA = E \oint dA = E(4\pi r^2) = \frac{q}{\epsilon_0}$$

where we have used the fact that the surface area of a sphere is $4\pi r^2$. Now, we solve for the electric field:

$$E = \frac{q}{4\pi\epsilon_0 r^2} = k_e \frac{q}{r^2}$$

The point charge $q$ is at the center of the spherical gaussian surface, and **E** is parallel to $d$ **A** at every point on the surface

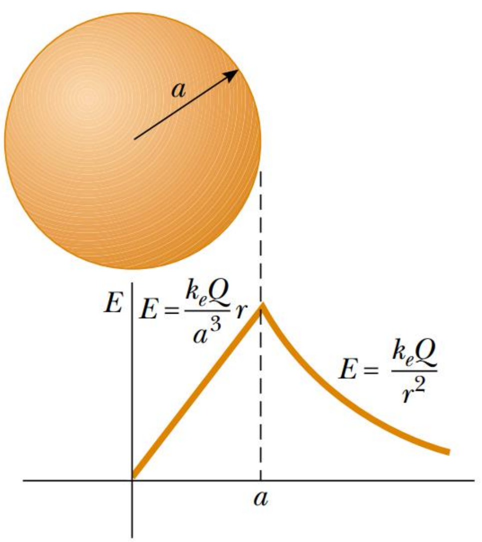
## Example 24.5 A Spherically Symmetric Charge Distribution

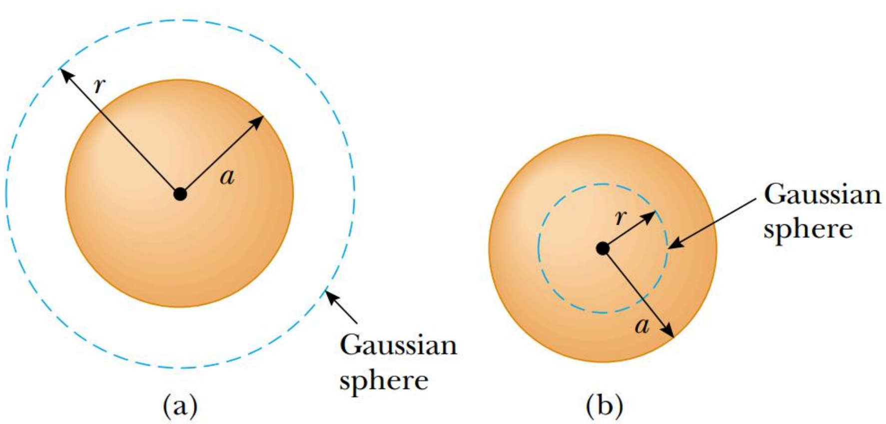
An insulating solid sphere of radius $a$ has a uniform volume charge density ρ and carries a total positive charge $Q$ .

**(A)** Calculate the magnitude of the electric field at a point outside the sphere.

**(B)** Find the magnitude of the electric field at a point inside the sphere.

**What If?** **Suppose we approach the radial position $r = a$ from inside the sphere and from outside. Do we measure the same value of the electric field from both directions?**




A uniformly charged insulating sphere of radius $a$ and total charge $Q$.

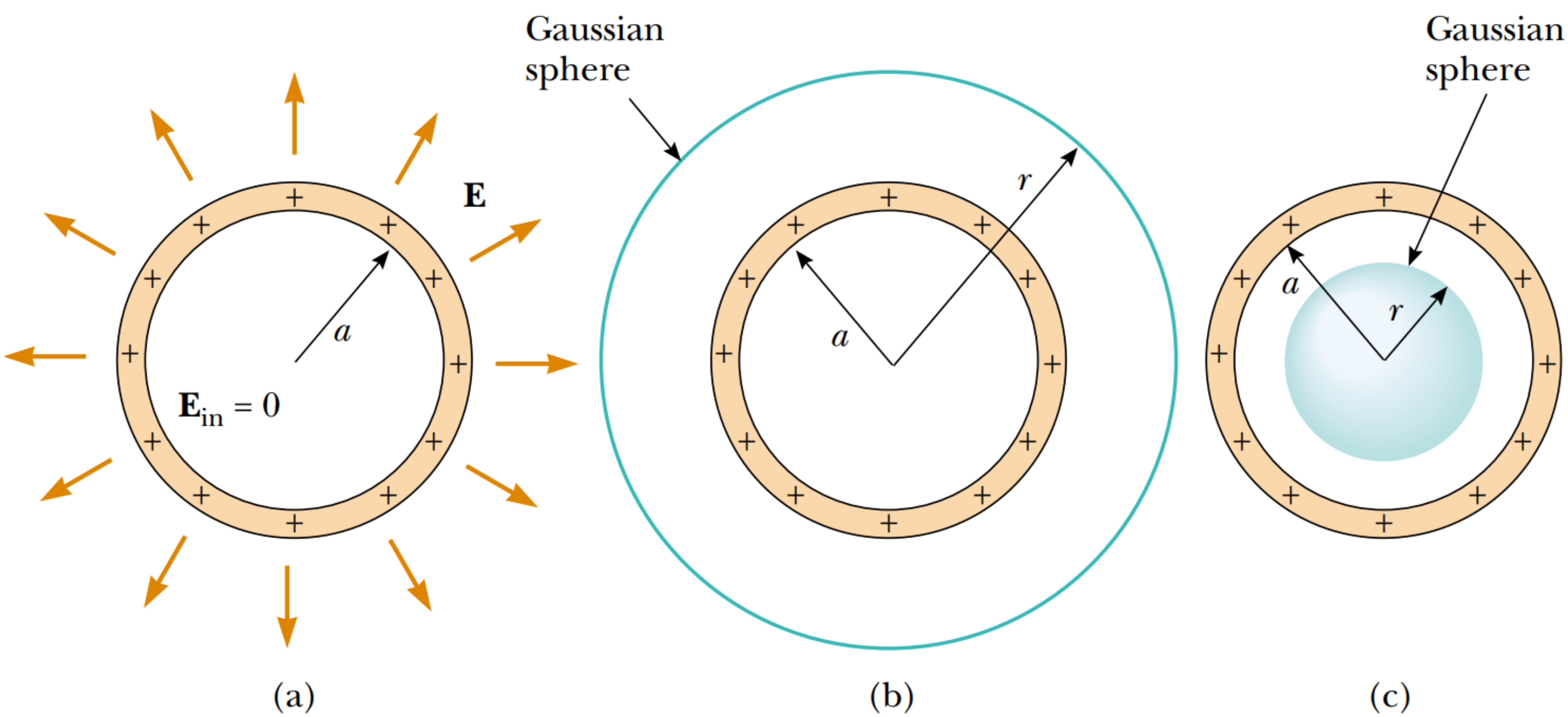
The electric field inside the sphere ($r < a$) varies linearly with $r$. The field outside the sphere ($r > a$) is the same as that of a point charge $Q$ located at $r = 0$.

## Example 24.6 The Electric Field Due to a Thin Spherical Shell

A thin spherical shell of radius $a$ has a total charge $Q$ distributed uniformly over its surface (Fig. 24.13a). Find the electric field at points

**(A)** outside and

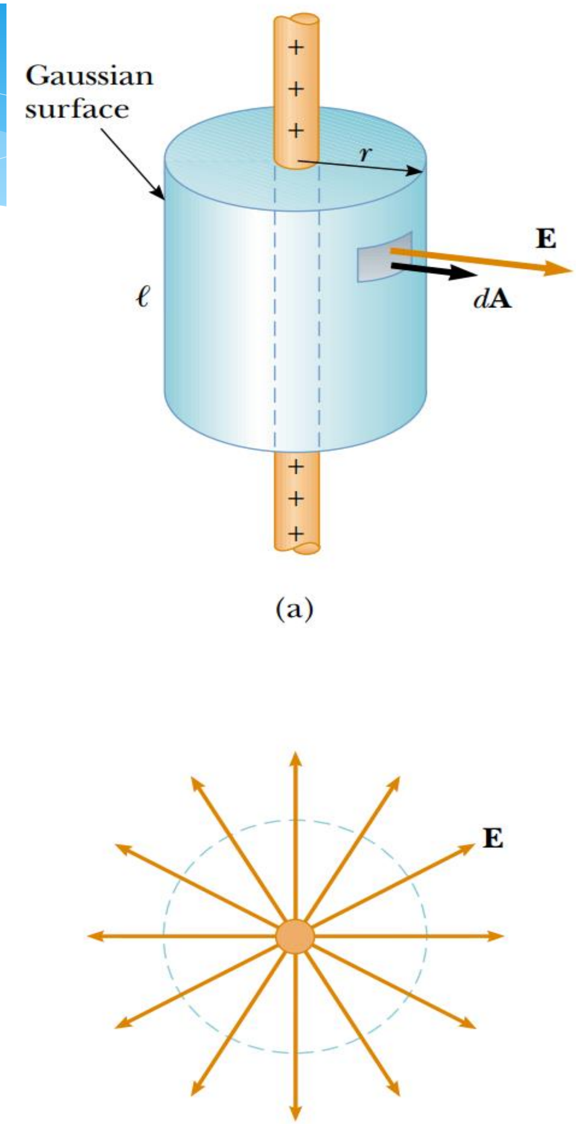
**(B)** inside the shell.




(a) The electric field inside a uniformly charged spherical shell is zero. The field outside is the same as that due to a point charge $Q$ located at the center of the shell. (b) Gaussian surface for $r > a$. (c) Gaussian surface for $r < a$.

# Example 24.7 A Cylindrically Symmetric Charge Distribution

Find the electric field a distance $r$ from a line of positive charge of infinite length and constant charge per unit length $\lambda$ (Fig. 24.14a).




(a) An infinite line of charge surrounded by a cylindrical gaussian surface concentric with the line. (b) An end view shows that the electric field at the cylindrical surface is constant in magnitude and perpendicular to the surface.

Cylindrical gaussian surface of radius $r$ and length $l$ that is coaxial with the line charge.

The flux through the ends of the gaussian cylinder is zero because **E** is parallel to these surfaces.

The total charge inside our gaussian surface is $\lambda\ell$. Applying Gauss's law and conditions (1) and (2), we find that for the curved surface

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = E \oint dA = EA = \frac{q_{\text{in}}}{\epsilon_0} = \frac{\lambda\ell}{\epsilon_0}$$

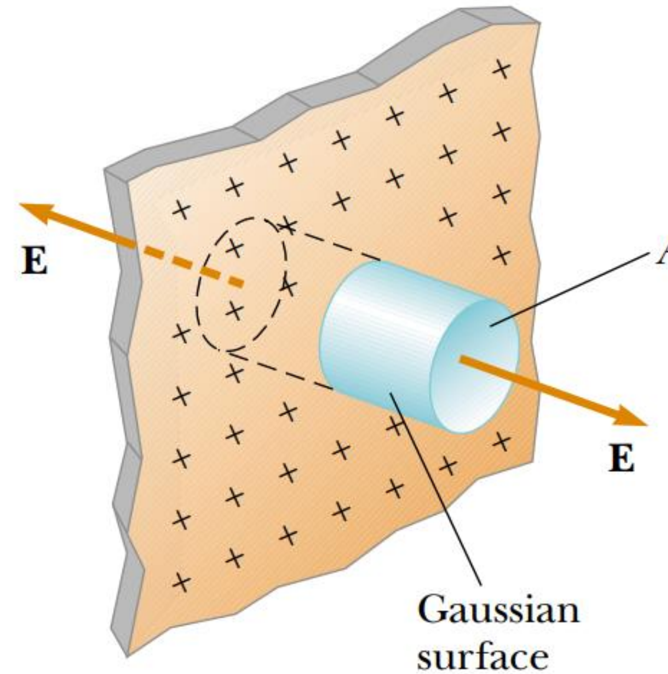
The area of the curved surface is $A = 2\pi r\ell$; therefore,

$$E(2\pi r\ell) = \frac{\lambda\ell}{\epsilon_0}$$

$$E = \frac{\lambda}{2\pi\epsilon_0 r} = 2k_e\frac{\lambda}{r} \qquad (24.7)$$

# Example 24.8 A Plane of Charge

Find the electric field due to an infinite plane of positive charge with uniform surface charge density $\sigma$.




The flux through each end of the cylinder is $EA$; hence, the total flux through the entire gaussian surface is just that through the ends, $\Phi_E = 2EA$.

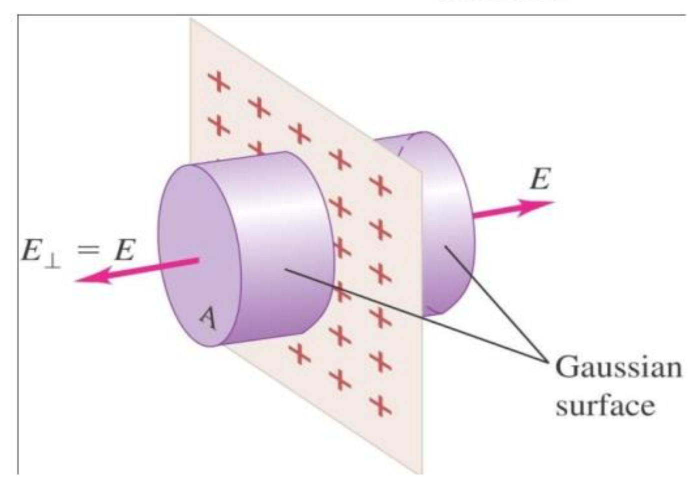
Noting that the total charge inside the surface is $q_{\text{in}} = \sigma A$, we use Gauss's law and find that the total flux through the gaussian surface is

$$\Phi_E = 2EA = \frac{q_{\text{in}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

leading to

$$\boxed{E = \frac{\sigma}{2\epsilon_0}} \qquad (24.8)$$

Because the distance from each flat end of the cylinder to the plane does not appear in Equation 24.8, we conclude that $E = \sigma/2\epsilon_0$ at *any* distance from the plane. That is, the field is uniform everywhere.




A cylindrical gaussian surface penetrating an infinite plane of charge. The flux is *EA* through each end of the gaussian surface and zero through its curved surface

## Table 24.1

**Typical Electric Field Calculations Using Gauss's Law**

| Charge Distribution | Electric Field | Location |
|---|---|---|
| Insulating sphere of radius $R$, uniform charge density, and total charge $Q$ | $\begin{cases} k_e \dfrac{Q}{r^2} \\ k_e \dfrac{Q}{R^2} r \end{cases}$ | $r > R$<br>$r < R$ |
| Thin spherical shell of radius $R$ and total charge $Q$ | $\begin{cases} k_e \dfrac{Q}{r^2} \\ 0 \end{cases}$ | $r > R$<br>$r < R$ |
| Line charge of infinite length and charge per unit length $\lambda$ | $2k_e \dfrac{\lambda}{r}$ | Outside the line |
| Infinite charged plane having surface charge density $\sigma$ | $\dfrac{\sigma}{2\epsilon_0}$ | Everywhere outside the plane |
| Conductor having surface charge density $\sigma$ | $\begin{cases} \dfrac{\sigma}{\epsilon_0} \\ 0 \end{cases}$ | Just outside the conductor<br>Inside the conductor |

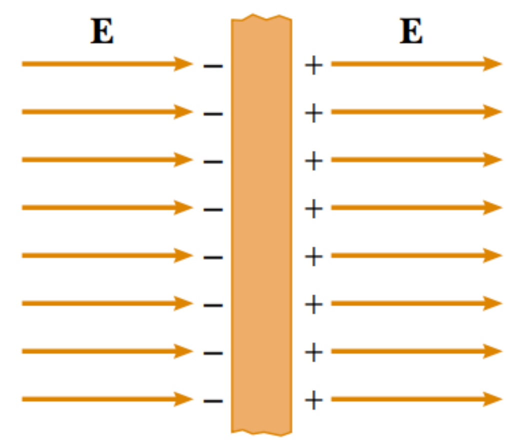
# 24.4 Conductors in Electrostatic Equilibrium

❖ ***Good electrical conductor*** contains charges (electrons) that are not bound to any atom and therefore are free to move about within the material.

❖ Conductor is ***in electrostatic equilibrium*** when there is no net motion of charge within a conductor, **. Charge is not moving**

## Properties of a conductor in electrostatic equilibrium

1. The electric field is zero everywhere inside the conductor.




A conducting slab in an external electric field **E**. The charges induced on the two surfaces of the slab produce an electric field that opposes the external field, giving a resultant field of zero inside the slab.

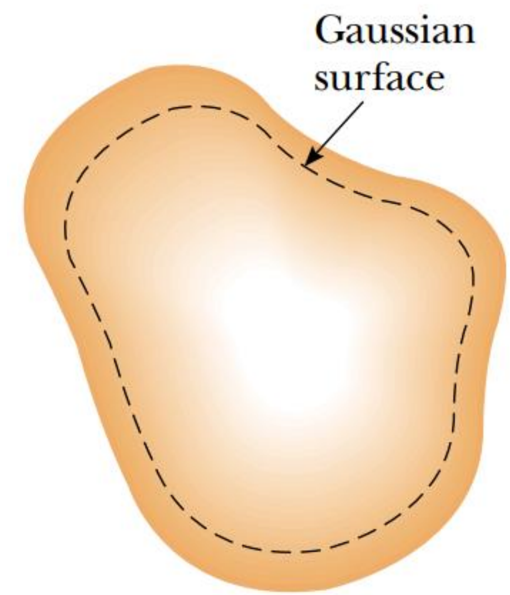
- If the field were not zero, free electrons in the conductor would experience an electric force(**F** = $q$ **E**) and would accelerate due to this force. This motion of electrons, however, would mean that the *conductor is not in electrostatic equilibrium.*

- Before the external field is applied, free electrons are uniformly distributed throughout the conductor. When the external field is applied, the free electrons accelerate to the left, causing a plane of negative charge to be present on the left surface. The movement of electrons to the left results in a plane of positive charge on the right surface. These planes of charge create an additional electric field inside the conductor that opposes the external field. As the electrons move, the surface charge densities on the left and right surfaces increase until the magnitude of the internal field equals that of the external field, resulting in a net field of zero inside the conductor.

## 24.4 Conductors in Electrostatic Equilibrium

2. If an isolated conductor carries a charge, the charge resides on its surface.

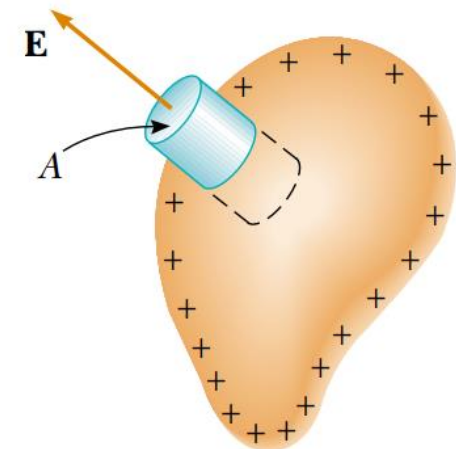
- Choose a gaussian surface inside but close to the actual surface.
- In electrostatic equilibrium. Therefore, the electric field must be zero at every point on the gaussian surface.
- The net flux through this gaussian surface is zero.
- Conclude that the net charge inside the gaussian surface is zero.




- **Any net charge on the conductor must reside on its surface.**

# 24.4 Conductors in Electrostatic Equilibrium

3. The electric field just outside a charged conductor is perpendicular to the surface of the conductor and has a magnitude $\sigma/\epsilon_0$, where $\sigma$ is the surface charge density at that point.




$$\Phi_E = \oint E\,dA = EA = \frac{q_{\text{in}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

$$q_{\text{in}} = \sigma A.$$

$$E = \frac{\sigma}{\epsilon_0}$$

A gaussian surface in the shape of a small cylinder is used to calculate the electric field just outside a charged conductor. The flux through the gaussian surface is *EA*. Remember that **E** is zero inside the conductor

4. On an irregularly shaped conductor, the surface charge density is greatest at locations where the radius of curvature of the surface is smallest.

س10) الفيض الكهربي Φ الكلي حول بروتون يساوي:

**Q10)** The total electric flux Φ around proton equals:

**A.** $1.6\times10^{-19}$ **B.** $1.4\times10^{-30}$ **C.** $18\times10^{-9}$ **D.** $55\times10^{6}$

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س11) كرة عازلة مصمته نصف قطرها $a$ ويتوزع بانتظام على حجمها شحنة مقدارها $Q$. مقدار المجال الكهربي عند نقطة تبعد مسافه $r$ من مركز الكرة بحيث $(r < a)$ يعطى بالعلاقه:

**Q11)** An insulator solid sphere of radius $a$ has a total positive charge $Q$ uniformly distributed throughout its volume. The magnitude of the electric field at a point at distance $r$ from the center of the sphere $(r < a)$ is given by the relation:

**A.** $\frac{k\,Q\,a}{r^3}$ **B.** $\frac{k\,Q\,r}{a^3}$ **C.** $\frac{k\,Q}{r^2}$ **D.** $\frac{k\,Q\,r^2}{a^3}$

**24.** A solid sphere of radius 40.0 cm has a total positive charge of 26.0 $\mu$C uniformly distributed throughout its volume. Calculate the magnitude of the electric field (a) 0 cm, (b) 10.0 cm, (c) 40.0 cm, and (d) 60.0 cm from the center of the sphere.

$\boxed{\textbf{31.}}$ Consider a thin spherical shell of radius 14.0 cm with a total charge of 32.0 $\mu$C distributed uniformly on its surface. Find the electric field (a) 10.0 cm and (b) 20.0 cm from the center of the charge distribution.

# Selected Solved Problems (Chapter # 24)

س12) كرة عازلة مصمته نصف قطرها 12 cm تحوي شحنة مقدارها 40 $\mu C$ موزعة بانتظام خلال حجمها. مقدار المجال الكهربي عند سطح الكرة يساوي:

**Q12)** An insulator solid sphere of radius 12 cm has a charge of 40 $\mu C$ uniformly distributed throughout its volume. The magnitude of the electric field at the sphere surface equals:

**A.** 3 MN/C **B.** 25 MN/C **C.** 9.4 MN/C **D.** Zero

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س13) في السؤال السابق (12) إذا كانت الكرة موصلة فان مقدار المجال الكهربي عند نقطة تبعد 10 cm من مركز الكرة يساوي:

**Q13)** In the previous question (12), if the sphere is conductor, the magnitude of the electric field at a point 10 cm from the center of the sphere equals:

**A.** Zero **B.** 20.8 MN/C **C.** 36 MN/C **D.** 0.5 MN/C

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س14) فتيل طويل جدا شحنته لوحدة الأطوال 50 nC/m فاذا كان مقدار المجال الكهربي له عند نقطة حول منتصفه هو 60 N/C فان بعد هذة النقطة من منتصف الفتيل يساوي:

**Q14)** A very long filament has charge per unit length 50 nC/m. If the electric field at a point around its middle is 60 N/C , the distance of the point from the filament equals:

**A.** 30 cm **B.** 25 cm **C.** 15 m **D.** 12 m

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س15) إذا كانت كثافة الشحنه السطحيه ( σ ) لشريحة لانهائية عازلة هي $35.4 \times 10^{-12}$ C/m$^2$ فان المجال الكهربي مباشرة فوق الشريحة يساوي:

**Q15)** If the surface charge density (σ) of an infinite insulator sheet is $35.4 \times 10^{-12}$ C/m$^2$ , the electric field just above the sheet equals:

**A.** 2 N/C **B.** 4 N/C **C.** 8.85 MN/C **D.** Zero

# Selected Solved Problems (Chapter # 24)

س8) إذا مُلئَ مكعب طول ضلعه 8 cm بشحنة كثافتها الحجمية $40\ nC/m^3$ فان الفيض الكهربي خلال أسطح المكعب يساوي:

$\rho = \frac{q}{V}$

**Q8)** If a cube of 8 cm edges is filled with a charge of uniform volume density of $40\ nC/m^3$, the total electric flux through the surfaces of the cube equals:

A. 2.9    B. 1.8    C. 2    D. 2.3

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س9) تحمل قشرة كرويه رقيقه نصف قطرها 16 cm شحنة 32 $\mu C$ موزعه بانتظام على سطحها. مقدار المجال الكهربي عند نقطه تبعد 10 cm من مركز الشريحه يساوي:

**Q9)** A thin spherical shell of radius 16 cm carry a total charge of 32 $\mu C$ distributed uniformly on its surface. The electric field at a point 10 cm from the center of the shell equals:

A. $7 \times 10^6$    B. $28.8 \times 10^6$    C. $46 \times 10^6$    D. Zero

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س10) اذا كان المجال الكهربي عند نقطة تبعد 18 mm من منتصف فتيل مستقيم طويل يساوي $9 \times 10^6$ N/C فان شحنة الفتيل لوحدة الأطوال λ تساوي:

**Q10)** If the electric field at a point of 18 mm from the center of a long straight filament is $9 \times 10^6$ N/C, the filament charge per unit length λ equals:

A. 9 C/m    B. 2 μC/m    C. 9 μC/m    D. 162 mC/m