1 Continuous random variable

Definition 1 *The function $f(x)$ is a probability density function (pdf) for the continuous random variable X, defined over the set of real numbers, if*

1. $f(x) \geq 0$, *for all* $x \in R$.

2. $\int_{-\infty}^{\infty} f(x)dx = 1$.

3. $\Pr(a < X < b) = \int_a^b f(x)dx$.

Example 2 *Suppose that the error in the reaction temperature, in $^\circ C$, for a controlled laboratory experiment is a continuous random variable X having the probability density function*

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & \textit{elsewhere} \end{cases}$$

(a) Verify that $f(x)$ is a density function.

(b) Find $\Pr(0 < X \leq 1)$.

Definition 3 *The cumulative distribution function* $F(x)$ *of a continuous random variable* X *with density function* $f(x)$ *is*

$$F(x) = \mathrm{Pr}(X \leq x) = \int_{-\infty}^{x} f(t)dt, \text{ for } -\infty < x < \infty.$$

Example 4 *For the density function of Example 2, find* $F(x)$*, and use it to evaluate* $\mathrm{Pr}(0 < X \leq 1)$*.*

Definition 5 (Mean of a Random Variable) *Let* X *be a random variable with probability distribution* $f(x)$*. The mean, or expected value, of* X *is*

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)dx$$

Example 6 *For the density function of Example 2, find* $E(X)$*.*

Theorem 7 *Let X be a random variable with probability distribution $f(x)$. The expected value of the random variable $g(X)$ is*

$$\mu_{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

Example 8 *Calculate the variance of $g(X) = 2X + 3$, where X is a random variable with probability distribution*

x	0	1	2	3
$f(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$

Theorem 9 (Variance of Random Variable) *Let X be a random variable with probability distribution $f(x)$ and mean μ. The variance of X is*

$$\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)$$

Theorem 10 *Let X a random variable. The variance of a random variable X is*

$$\sigma^2 = E(X^2) - E(X)^2.$$

Theorem 11 *Let X a random variable. If a and b are constants, then $E(aX + b) = aE(X) + b$.*

Theorem 12 *The expected value of the sum or difference of two or more functions of a random variable X is the sum or difference of the expected values of the functions. That is,*

$$E[g(X) \pm h(X)] = E[g(X)] \pm E[h(X)].$$

Example 13 *Let X be a random variable with probability distribution as follows:*

x	0	1	2	3
$f(x)$	$\frac{1}{3}$	$\frac{1}{2}$	0	$\frac{1}{6}$

Find the expected value of $Y = (X - 1)^2$.

Definition 14 (Discrete Uniform Random Variable) *A random variable X is called discrete uniform if it has a finite number of possible values, say $x_1, x_2, ..., x_n$, and* $\Pr(X = x_i) = 1/n$ *for all i.*

Definition 15 (Normal Distribution) *The density of the normal random variable X, with mean μ and variance σ^2, is*

$$\Pr(X = x) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-1}{2\sigma^2}(x-\mu)^2}, \quad -\infty < x < \infty,$$

where $\pi = 3.14159 \ldots$ and $e = 2.71828 \ldots$.

Theorem 16 *The mean and variance of $N(\mu, \sigma)$ are μ and σ^2, respectively. Hence, the standard deviation is σ.*

Definition 17 *The distribution of a normal random variable with mean 0 and variance 1 is called a standard normal distribution Z.*

The exponential random variable is used when we are interested in the time of the first arrival or the time between arrival.

Definition 18 *The continuous random variable X has an exponential distribution, with parameter λ, if its density function is given by* $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0 & \text{elsewhere} \end{cases}$ *where $\lambda > 0$.*

Theorem 19 *The mean and variance of the exponential distribution are $\mu = 1/\lambda$ and $\sigma^2 = 1/\lambda^2$.*

If X is the time of arrival of the first customer and if the average time is 30 minutes, then $\lambda = 1/30$.

Example 20 *Suppose that a system contains a certain type of component whose time, in years, to failure is given by T. The random variable T is modeled nicely by the exponential distribution with mean time to failure is 5.*

1- If one component is installed, what is the probability that it is still functioning at the end of 8 *years?*
2- If 5 *of these components are installed in different systems, what is the probability that at least* 2 *are still functioning at the end of* 8 *years? (Hint: use the binomial distribution)*