1 Discrete random variable

Definition 1 *The set of ordered pairs* $(x, f(x))$ *is a probability function, probability mass function, or probability distribution of the discrete random variable* X *if, for each possible outcome* x*,*
1. $f(x) \geq 0$,
2. $\sum_x f(x) = 1$,
3. $P(X = x) = f(x)$.

Definition 2 *The cumulative distribution function* $F(x)$ *of a discrete random variable* X *with probability distribution* $f(x)$ *is*

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t), for - \infty < x < \infty$$

Definition 3 (Mean of a Random Variable) *Let* X *be a random variable with probability distribution* $f(x)$*. The mean, or expected value, of* X *is*

$$\mu = E(X) = \sum_x x f(x)$$

Example 4 *A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.*

Example 5 *Let X represent the number of good components in the sample. The probability distribution of X is f(x) =*$\dfrac{\binom{4}{x}\binom{3}{3-x}}{\binom{N}{n}}$, $x = 0, 1, 2, 3$.

Simple calculations yield $f(0) = 1/35$, $f(1) = 12/35$, $f(2) = 18/35$, *and* $f(3) = 4/35$. *Therefore,*

$$\mu = E(X) = (0)\frac{1}{35} + (1)\frac{12}{35} + (2)\frac{18}{35} + (3)\frac{4}{35} = 12/7 = 1.7$$

Thus, if a sample of size 3 is selected at random over and over again from a lot of 4 good components and 3 defective components, it will contain, on average, 1.7 good components.

Theorem 6 *Let X be a random variable with probability distribution $f(x)$. The expected value of the random variable $g(X)$ is*

$$\mu_{g(X)} = E[g(X)] = \sum_x g(x)f(x)$$

Example 7 *Suppose that the number of cars X that pass through a car wash between 4:00 P.M. and 5:00 P.M. on any sunny Friday has the following probability distribution:*

x	4	5	6	7	8	9
$f(x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

Let $g(X) = 2X - 1$ represent the amount of money, in dollars, paid to the attendant by the manager. Find the attendant's expected earnings for this particular time period.

Theorem 8 (Variance of Random Variable) *Let X be a random variable with probability distribution $f(x)$ and mean μ. The variance of X is*

$$\sigma^2 = E[(X - \mu)^2] = \sum_x (x - \mu)^2 f(x)$$

The positive square root of the variance, ■, is called the standard deviation of X.

Definition 9 (Bernouilli Process) *Strictly speaking, the Bernoulli process must possess the following properties:*
1. The experiment consists of repeated trials.
2. Each trial results in an outcome that may be classified as a success or a failure.
3. The probability of success, denoted by p, remains constant from trial to trial.
4. The repeated trials are independent.

Definition 10 (Binomial Distribution) *A Bernoulli trial can result in a success with probability p and a failure withprobability $q = 1 - p$. Then the probability distribution of the binomial random variable X, the number of successes in n independent trials, is*

$$\text{Pr}(X = x) = \begin{pmatrix} n \\ x \end{pmatrix} p^x q^{n-x},\ x = 0, 1, 2, ..., n.$$

Theorem 11 *The mean and variance of the binomial distribution $B(n, p)$ are*

$$\mu = np \text{ and } \sigma^2 = npq.$$

Definition 12 (HypergeometricDistribution) *The probability distribution of the hypergeometric random variable X, the numberof successes in a random sample of size n selected from N items of which K are labeled success and $N - K$ labeled failure, is*

$$\Pr(X = x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

Theorem 13 *The mean and variance of the hypergeometric distribution $h(N, K, n)$ are*

$$\mu = n\frac{K}{N} \text{ and } \sigma^2 = \frac{N-n}{N-1} \cdot \frac{nK}{N}\left(1 - \frac{nK}{N}\right).$$

Theorem 14 (Approximation) *If n is small compared to N, then a binomial distribution $B(n, p = K/N)$ can be used to approximate the hypergeometric distribution $h(N, K, n)$.*

Definition 15 *Let X the number of outcomes occurring during a given time interval. X is called a Poisson random variable when its probability distribution is given by*

$$\Pr(X = x) = e^{-\lambda}\frac{\lambda^x}{x!},\ x = 0, 1, 2, ...,$$

where ■ is the average number of outcomes.

Theorem 16 *Both the mean and the variance of the Poisson distribution $P(\lambda)$ are λ.*

Theorem 17 (Approximation) *Let X be a binomial random variable with probability distribution $B(n, p)$. When n is large ($n \to \infty$), and p small ($p \to 0$), then the poisson distribution can be used to approximate the binomial distribtion $B(n, p)$ by taking $\lambda = np$.*