Holomorphic Functions

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Definition

Let $f:\ \Omega \longrightarrow \mathbb{C}$ *be a function defined on a non empty open set* Ω.

1. *We say that* f *is differentiable at* $z \in \Omega$ *if there exists* $\ell \in \mathbb{C}$ *such that*

$$\lim_{z \to a} \frac{f(w) - f(z)}{w - z} = \lim_{h \to 0, h \in \mathbb{C}^*} \frac{f(z+h) - f(z)}{h} = \ell.$$

   We denote $\ell = f'(z)$ *and called the derivative of* f *at* z.

2. *We say that* f *is holomorphic on* Ω *if* f *is differentiable at any point of* Ω.
   We denote $\mathcal{H}(\Omega)$ *the set of holomorphic functions on* Ω.

Examples

1. *The function* $f(z) = z^n$ *is holomorphic on* $\mathbb{C}$*, for every* $n \in \mathbb{N}$*.*
2. *The function* $f(z) = \bar{z}$ *is not differentiable at any point of* $\mathbb{C}$ *because* $\dfrac{\overline{z+h} - \bar{z}}{h} = \dfrac{\bar{h}}{h}$ *and* $\lim\limits_{h\to 0} \dfrac{\bar{h}}{h}$ *does not exit.*
3. *If* $f : \Omega \longrightarrow \mathbb{C}$ *is a holomorphic function, then the function* f^* *defined by* $f^*(z) = \overline{f(\bar{z})}$ *is holomorphic on* $\overline{\Omega} = \{\bar{z};\ z \in \Omega\}$*, indeed If* $z, a \in \overline{\Omega}$*,*

$$\lim_{z\to a} \frac{f^*(z) - f^*(a)}{z - a} = \lim_{z\to a} \overline{\left(\frac{f(\bar{z}) - f(\bar{a})}{\bar{z} - \bar{a}}\right)} = \overline{f'(\bar{a})}.$$

Proposition (Exercise)

1. *If f and g are holomorphic on Ω, then $f+g$, fg are also holomorphic on Ω.*
   The function $\frac{f}{g}$ is holomorphic on the open set where g does not vanishes.
2. *If $f:\Omega_1 \longrightarrow \mathbb{C}$ is a holomorphic function and $g:\Omega_2 \longrightarrow \mathbb{C}$ is a holomorphic function such that $g(\Omega_2)\subset\Omega_1$, then $f\circ g$ is holomorphic on Ω_2 and $(f\circ g)'(z)=f'(g(z))g'(z)$.*

Theorem (Cauchy-Riemann conditions)

Let $f(z) = U(x, y) + \mathrm{i}V(x, y)$ be a function defined on a neighborhood of $z_0 = x_0 + \mathrm{i}y_0$. ($U = \Re f$ and $V = \Im f$). We assume that the functions U and V are differentiable at (x_0, y_0). Then the function f of complex variable $z = x + \mathrm{i}y$ is differentiable at z_0, if and only if

$$\begin{cases} \dfrac{\partial U}{\partial x}(x_0, y_0) = \dfrac{\partial V}{\partial y}(x_0, y_0) \\ \\ \dfrac{\partial U}{\partial y}(x_0, y_0) = -\dfrac{\partial V}{\partial x}(x_0, y_0) \end{cases}. \qquad (1)$$

These conditions are called the Cauchy-Riemann conditions. They are equivalent to the following condition

$$\frac{\partial f}{\partial y}(z_0) = \mathrm{i}\frac{\partial f}{\partial x}(z_0). \tag{2}$$

Proof

Necessary condition

If f is differentiable at $z_0 = x_0 + \mathrm{i}y_0$, then $\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = f'(z_0)$. If we take the limit when h tends to 0, h real, we have

$$\begin{aligned} f'(z_0) &= \lim_{h\to 0} \frac{U(x_0+h,y_0)-U(x_0,y_0)}{h} + \mathrm{i}\frac{V(x_0+h,y_0)-V(x_0,y_0)}{h} \\ &= \frac{\partial U}{\partial x}(x_0,y_0) + \mathrm{i}\frac{\partial V}{\partial x}(x_0,y_0) = \frac{\partial f}{\partial x}(x_0+\mathrm{i}y_0). \end{aligned} \tag{3}$$

If we take the limit when $h = \mathrm{i}t$, with t real, we have

$$\begin{aligned} f'(z_0) &= \lim_{t\to 0} \frac{U(x_0, y_0+t) - U(x_0, y_0)}{h} + \mathrm{i}\frac{V(x_0, y_0+t) - V(x_0, y_0)}{\mathrm{i}t} \\ &= -\mathrm{i}\frac{\partial U}{\partial y}(x_0, y_0) + \frac{\partial V}{\partial y}(x_0, y_0) = -\mathrm{i}\frac{\partial f}{\partial y}(z_0). \qquad (4) \end{aligned}$$

We have then the Cauchy-Riemann conditions.

Sufficient condition

By definition of the differentiability of function of two real variables, we have

$$\begin{cases} U(x_0+s, y_0+t) - U(x_0, y_0) = s\dfrac{\partial U}{\partial x}(x_0, y_0) + t\dfrac{\partial U}{\partial y}(x_0, y_0) + |h|\varepsilon_1(h) \\ V(x_0+s, y_0+t) - V(x_0, y_0) = s\dfrac{\partial V}{\partial x}(x_0, y_0) + t\dfrac{\partial V}{\partial y}(x_0, y_0) + |h|\varepsilon_2(h) \end{cases}$$

with $h = s + \mathrm{i}t$, $\lim_{h\to 0} \varepsilon_1(h) = \lim_{h\to 0} \varepsilon_2(h) = 0$. Thus

$$f(z_0+h)-f(z_0) = s\frac{\partial U}{\partial x}(x_0, y_0)+t\frac{\partial U}{\partial y}(x_0, y_0)+\mathrm{i}\Big(s\frac{\partial V}{\partial x}(x_0, y_0)+t\frac{\partial V}{\partial y}(x_0, y_0$$

with $\varepsilon = \varepsilon_1 + \mathrm{i}\varepsilon_2$.

It follows by Cauchy-Riemann conditions that

$$f(z_0+h)-f(z_0)=Ah+|h|\eta(h),$$

with $A=\dfrac{\partial U}{\partial x}(x_0,y_0)+\mathrm{i}\dfrac{\partial V}{\partial x}(x_0,y_0)$ and $\eta(h)$ tends to 0 when h tends to 0. Thus f is differentiable at z_0 and $f'(z_0)=A$.

Remark (Riemann 1851)

If U and V are two functions twice continuously differentiable on an open subset Ω of $\mathbb{C}$ and if $f = U + \mathrm{i}V$ is holomorphic on Ω, then $\Delta U = \Delta V = 0$, with $\Delta U = \dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2}$. ($\Delta$ is called the Laplace operator). We say that U and V are harmonic.

Corollary

If f is holomorphic on Ω, then

$$\begin{aligned} f'(z) &= \frac{\partial f}{\partial x}(x,y) = -\mathrm{i}\frac{\partial f}{\partial y}(x,y) \\ &= \frac{\partial U}{\partial x}(x,y) + \mathrm{i}\frac{\partial V}{\partial x}(x,y) = \frac{\partial V}{\partial y}(x,y) - \mathrm{i}\frac{\partial U}{\partial y}(x,y) \\ &= \frac{\partial U}{\partial x}(x,y) - \mathrm{i}\frac{\partial U}{\partial y}(x,y) = \frac{\partial V}{\partial y}(x,y) + \mathrm{i}\frac{\partial V}{\partial x}(x,y), \end{aligned}$$

where $z = x + \mathrm{i}y$.

Example

Determine the holomorphic functions $f : \mathbb{C} \longrightarrow \mathbb{C}$ *such that* $\Re f(z) = U(x, y) = x^2 - y^2 - 2xy$.
Let $V = \Im f$, *then by Cauchy-Riemann conditions,* $\dfrac{\partial V}{\partial y} = 2x - 2y$,
thus $V = 2xy - y^2 + g(x)$. *Furthermore*
$\dfrac{\partial V}{\partial x} = 2y + g'(x) = -(-2y - 2x) = 2x + 2y$, *thus*
$V = x^2 - y^2 + 2xy + C$ *and* $f(z) = (1 + \mathrm{i})z^2 + \mathrm{i}C$.

Example

Determine the holomorphic functions $f : \mathbb{C} \longrightarrow \mathbb{C}$ *such that* $\Re f(z) = x^3 - 3xy^2 - 2xy - 1$.
If $V = \Im f$, *then* $\dfrac{\partial V}{\partial y} = 3x^2 - 3y^2 - 2y$, *thus* $V = 3x^2y - y^3 - y^2 + g(x)$. *Furthermore* $\dfrac{\partial V}{\partial x} = 6xy + g'(x) = 6xy + 2x$, *thus* $V = 3x^2y - y^3 - y^2 + x^2 + C$.
$f'(z) = \dfrac{\partial f}{\partial x} = 3x^2 - 3y^2 - 2y + \mathrm{i}(6xy + 2x) = 3(x^2 - y^2 + 2\mathrm{i}xy) + 2\mathrm{i}(x + \mathrm{i}y) = 3z^2 + 2\mathrm{i}z$. *Thus* $f(z) = z^3 + \mathrm{i}z^2 + 1 + \mathrm{i}c$, *where* $c \in \mathbb{R}$.

Proposition

Let f be a holomorphic function on a domain Ω. Then $f' \equiv 0$ on $\Omega \iff f$ is constant on Ω.

Proof

The sufficient condition is trivial. For the necessary condition, it suffices to show that f is locally constant. Let $z_0 \in \Omega$ and $r > 0$ such that $D(z_0, r) \subset \Omega$. Let $z_1 \in D(z_0, r)$, the complex number $z_2 = \Re z_1 + \mathrm{i}\Im z_0 \in D(z_0, r)$ and $f(z_0) = f(z_2)$ because $\dfrac{\partial f}{\partial x} = 0$ and $f(z_2) = f(z_1)$ because $\dfrac{\partial f}{\partial y} = 0$, thus $f(z_0) = f(z_1)$.

Proposition

Let $f : \Omega \longrightarrow \mathbb{C}$ be a holomorphic function on a domain Ω of $\mathbb{C}$. Then the following properties are equivalent.

1. *f is constant on Ω.*
2. *$\Re f$ is constant on Ω.*
3. *$\Im f$ is constant on Ω.*
4. *$|f|$ is constant on Ω.*
5. *$\bar{f}$ is holomorphic on Ω.*

Proof

• It is obvious that 1) $\Rightarrow$ 2).

• By The Cauchy-Riemann conditions (1), 2) $\iff$ 3).
Since 2) $\iff$ 3), then 3) $\Rightarrow$ 4).

• If $|f| = 0$, then $\bar{f}$ is holomorphic on Ω.
If $|f| = c \neq 0$, then $f\bar{f} = c$ and $\bar{f} = \dfrac{c^2}{f}$ is holomorphic on Ω.

• $\bar{f}$ is holomorphic on Ω. In use the Cauchy-Riemann conditions for f and $\bar{f}$, we find f is constant.

$\square$

Theorem

Let f be the holomorphic function defined by the power series $\sum_{n\geq 0} a_n z^n$ which admits $R > 0$ as radius of convergence, then the function g defined by the power series $\sum_{n\geq 1} n a_n z^{n-1}$ admits R as radius of convergence. The function f is holomorphic on $D(0, R)$ and $f'(z) = g(z)$.

For the proof of this theorem, we need the following lemma

Lemma

Let $z \in \mathbb{C}$ *and* $h \in \mathbb{C}$ *such that* $0 < |h| \leq r$*, then for all* $n \in \mathbb{N}^*$

$$|(z+h)^n - z^n - nhz^{n-1}| \leq \frac{|h|^2}{r^2}(|z|+r)^n \tag{5}$$

and

$$n|z|^{n-1} \leq \frac{1}{r}\big(2(|z|+r)^n + |z|^n\big) \tag{6}$$

Proof

By inequality (5)

$$\begin{aligned}
\left|(z+h)^n - z^n - nhz^{n-1}\right| &= \left|\sum_{k=0}^{n} C_n^k h^k z^{n-k} - z^n - nhz^{n-1}\right| \\
&= \left|\sum_{k=2}^{n} C_n^k h^k z^{n-k}\right| \\
&\leq |h|^2 \sum_{k=2}^{n} C_n^k |z|^{n-k} |h|^{k-2} \\
&\leq \frac{|h|^2}{r^2} \sum_{k=2}^{n} C_n^k |z|^{n-k} r^k \leq \frac{|h|^2}{r^2}(|z|+r)^n.
\end{aligned}$$

We have $|(z+h)^n - z^n - nhz^{n-1}| \geq nr|z|^{n-1} - |z|^n - (|z|+r)^n$.
By (5), we deduce

$$nr|z|^{n-1} \leq |z|^n+(|z|+r)^n+|(z+r)^n-z^n-nrz^{n-1}| \leq |z|^n+2(|z|+r)^n.$$

Proof of theorem 2.1

We denote R' the radius of convergence of the power series $\sum\limits_{n\geq 1} na_n z^{n-1}$. It is obvious that $R' \leq R$. Let $r > 0$ such that $|z| + r < R$. By lemma 2.2, we have $|na_n z^{n-1}| \leq \frac{1}{r}\big(2|a_n|(|z| + r)^n + |a_n||z|^n\big)$ and thus $\sum\limits_{n\geq 1} na_n z^{n-1}$ converges absolutely on $D(0, R)$. Thus the radius of convergence of the series defining g is greater than R. Thus $R = R'$.

By inequality (5),

$$\left|\frac{f(z+h)-f(z)}{h}-g(z)\right| \leq \frac{|h|}{r}\sum_{n=1}^{+\infty}|a_n|(|z|+r)^n,$$

this proves that when h tends to 0, $f'(z)=g(z)$, for all $z\in D(0,R)$.

Corollary

If $f(z) = \sum_{n=0}^{+\infty} a_n z^n$, *then f is infinitely continuously differentiable on* $D(0, R)$, $a_n = \frac{f^{(n)}(0)}{n!}$ *and* $f(z) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} z^n$. *(This series is called the Taylor's series of f at* 0*.)*

The exponential function e^z is defined by the series $\sum_{n\geq 0} \frac{z^n}{n!}$, ($e^z = \sum_{n=0}^{+\infty} \frac{z^n}{n!}$) and it fulfills the following properties

- $(\mathrm{e}^z)' = \mathrm{e}^z$,
- $\mathrm{e}^{z+w} = \mathrm{e}^z\mathrm{e}^w$ this results by definition of the product series,
- $\mathrm{e}^z\mathrm{e}^{-z} = 1$ for all $z \in \mathbb{C}$,
- $\mathrm{e}^x > 0$ for all $x \in \mathbb{R}$,
- $0 < \mathrm{e}^x < 1$ for all $x \in \mathbb{R}^*_-$,
- $\mathrm{e}^{\bar{z}} = \overline{\mathrm{e}^z}$,
- $|\mathrm{e}^{\mathrm{i}y}| = 1$ for all $y \in \mathbb{R}$, thus $|\mathrm{e}^{x+\mathrm{i}y}| = \mathrm{e}^x$ for all $(x, y) \in \mathbb{R}^2$.
- $\mathrm{e}^z \neq 0,\ \forall z \in \mathbb{C}$.

These properties prove that the exponential function $z \longmapsto \mathrm{e}^z$ $(\mathbb{C}, +) \longrightarrow (\mathbb{C}^*, \times)$ is an homomorphism of groups.

We define

- $\cos z = \dfrac{\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}}{2}$ the cosine function,
- $\sin z = \frac{\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}}{2i}$ the sine function,
- $\cosh z = \frac{\mathrm{e}^{z} + \mathrm{e}^{-z}}{2}$ the hyperbolic cosine function,
- $\sinh z = \frac{\mathrm{e}^{z} + \mathrm{e}^{-z}}{2}$ the hyperbolic sine function.

Thus $\mathrm{e}^{\mathrm{i}z} = \cos z + \mathrm{i} \sin z$, $\mathrm{e}^{x+\mathrm{i}y} = \mathrm{e}^{x}(\cos y + \mathrm{i} \sin y)$, y is an argument of e^{z} (this yields the Euler formula $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i} \sin\theta$).

Properties

$\cos z + \mathrm{i}\sin z = e^{\mathrm{i}z}$, $\cos z - \mathrm{i}\sin z = e^{-\mathrm{i}z}$, *then* $\cos^2 z + \sin^2 z = 1$.
$\cosh z + \sinh z = e^{z}$, $\cosh z - \sinh z = e^{-z}$, *then*
$\cosh^2 z - \sinh^2 z = 1$.
$\cosh(\mathrm{i}z) = \cos z$ *and* $\sinh(\mathrm{i}z) = \mathrm{i}\sin z$.

Proposition

The function $\mathrm{e}^{\mathrm{i}z}$ *is periodic. We denote* 2π *the period of* $\mathrm{e}^{\mathrm{i}z}$*.*

Proof

$\mathrm{e}^{z} = \mathrm{e}^{z+w} \iff \mathrm{e}^{w} = 1$, thus $w = \mathrm{i}y$, with $y \in \mathbb{R}$.

Since $\cos^2 y + \sin^2 y = 1$, then $(\sin y)' = \cos y \leq 1$.

Since $\sin 0 = 0$, then $\sin y \leq y$ for $y \geq 0$. In the same time, $(\cos y)' = -\sin y \geq -y$ and $\cos 0 = 1$, then $\cos y \geq 1 - \frac{y^2}{2}$. It follows that $\sin y \geq y - \frac{y^3}{3!}$ and $\cos y \leq 1 - \frac{y^2}{2} + \frac{y^4}{4!}$. These inequalities leads that $\cos\sqrt{3} < 0$, thus there exists $0 < y_0 < \sqrt{3}$ such that $\cos y_0 = 0$ and $\sin y_0 = \pm 1$. $\mathrm{e}^{\mathrm{i}y_0} = \pm\mathrm{i}$ and $\mathrm{e}^{4\mathrm{i}y_0} = 1$. Thus $4y_0$ is a period of the function $\mathrm{e}^{\mathrm{i}z}$ and it is the smallest period. Indeed, let $y_0 > y > 0$, $\sin y \geq y(1 - \dfrac{y^2}{6}) \geq \dfrac{y}{2} > 0$. This shows that the function $\cos y$ is strictly decreasing and $\sin y$ is strictly increasing on the interval $[0, y_0]$, thus $\sin y < \sin y_0 = 1$, $0 < \sin y < 1$, this yields that $\mathrm{e}^{\mathrm{i}y} \neq \pm 1$, $\mathrm{e}^{\mathrm{i}y} \neq \pm\mathrm{i}$ and $\mathrm{e}^{4\mathrm{i}y} \neq 1$. It follows that $4y_0$ is the smallest period denoted w_0. Let w be a period such that $nw_0 \leq w < (n+1)w_0$.

If $w \neq nw_0$, then $w - nw_0$ is a period and $0 < w - nw_0 < w_0$, this is impossible, thus $w = nw_0$. We denote $w_0 = 2\pi$, thus $\mathrm{e}^{\mathrm{i}\frac{\pi}{2}} = \mathrm{i}$, $\mathrm{e}^{\mathrm{i}\pi} = -1$ and $\mathrm{e}^{2\mathrm{i}\pi} = 1$.

Lemma

For all $z = x + \mathrm{i}y \in \mathbb{C} \setminus \{0\}$*, there exists* $r > 0$ *and a unique* $\theta \in [0, 2\pi[$ *such that* $z = r(\cos\theta + \mathrm{i}\sin\theta) = re^{\mathrm{i}\theta}$*.*

Proof

One has $\cos^2\theta + \sin^2\theta = 1$, thus $r = \sqrt{x^2 + y^2}$. We assume that $r = 1$.

• If $0 \leq x < 1$ and $y \geq 0$, we know that the function $f(\theta) = \cos\theta$ is decreasing and continuous of the interval $[0, \frac{\pi}{2}]$ with values on the interval $[0, 1]$ and as $x \in [0, 1[$, there exists a unique $\theta \in]0, \frac{\pi}{2}]$ such that $x = \cos\theta$, furthermore $x^2 + y^2 = 1$ and $y \geq 0$, thus $y = \sqrt{1 - x^2} = \sin\theta$, where $x + \mathrm{i}y = \cos\theta + \mathrm{i}\sin\theta$.

• If $x \geq 0$ and $y \leq 0$, $x \neq 1$. From which above, there exists a unique $\theta \in]0, \frac{\pi}{2}]$ such that $x - \mathrm{i}y = \cos\theta + \mathrm{i}\sin\theta$, where $\cos\theta - \mathrm{i}\sin\theta = x + \mathrm{i}y = \cos(2\pi - \theta) + \mathrm{i}\sin(2\pi - \theta)$ and $2\pi - \theta \in [\frac{3\pi}{2}, 2\pi[$.

• If $x \leq 0$, $x \neq -1$ and $y \geq 0$. From which above, there exists a unique $\theta \in]\frac{3\pi}{2}, 2\pi]$ such that $-x - \mathrm{i}y = \cos\theta + \mathrm{i}\sin\theta$, this yields that $x + \mathrm{i}y = \cos(-\pi + \theta) + \mathrm{i}\sin(-\pi + \theta)$, where $\theta - \pi \in [\frac{\pi}{2}, \pi[$.

• If $x \leq 0$, $x \neq 1$ and $y \leq 0$. From above, there exists a unique $\theta \in]0, \frac{\pi}{2}]$ such that $-x - \mathrm{i}y = \cos\theta + \mathrm{i}\sin\theta$, this yields that $x + \mathrm{i}y = \cos(\pi + \theta) + \mathrm{i}\sin(\pi + \theta)$, where $\theta + \pi \in]\pi, \frac{3\pi}{2}]$.
• If $x = 1$ and $y = 0$, then $x + \mathrm{i}y = \cos 0 + \mathrm{i}\sin 0$.
• If $x = -1$ and $y = 0$, then $x + \mathrm{i}y = \cos\pi + \mathrm{i}\sin\pi$.
The uniqueness of θ in $[0, 2\pi[$ results from the fact that 2π is the smallest period of the function $\mathrm{e}^{\mathrm{i}z}$. □

We will prove that the branches of the logarithm defined as above are holomorphic.

Remark

From which above, for all $\alpha \in \mathbb{R}$ *and all* $z \in \mathbb{C} \setminus \{0\}$*, there exists a unique* $r > 0$ *and a unique* $\theta \in [\alpha, \alpha + 2\pi[$ *such that* $z = r(\cos\theta + \mathrm{i}\sin\theta)$.

Proposition

The mapping $A\colon \mathbb{C}\setminus\mathbb{R}^+ \longrightarrow]0,2\pi[$ *defined by* $A(z) = A(r(\cos\theta + \mathrm{i}\sin\theta)) = \theta$ *is continuous.*

Proof

$z = r(\cos\theta + \mathrm{i}\sin\theta) = x + \mathrm{i}y$, where $\theta \in]0,2\pi[$.

$$x = r\cos\theta = -r\cos(\pi-\theta) = -2r\cos^2(\frac{\pi-\theta}{2}) + r.$$

$$y = r\sin\theta = r\sin(\pi-\theta) = 2r\cos(\frac{\pi-\theta}{2})\sin(\frac{\pi-\theta}{2}).$$

$$r - x = 2r\cos^2(\frac{\pi-\theta}{2}).$$

$$y = 2r\cos(\frac{\pi-\theta}{2})\sin(\frac{\pi-\theta}{2})$$

$$\frac{y}{-x+\sqrt{x^2+y^2}} = \mathrm{tg}(\frac{\pi-\theta}{2}) \Rightarrow \frac{\pi-\theta}{2} = \tan^{-1}(\frac{y}{\sqrt{x^2+y^2}-x}).$$

Thus

$$\theta = \pi - 2\tan^{-1}\left(\frac{y}{\sqrt{x^2+y^2}-x}\right),$$

and A is continuous.
In the same way, the mapping $\mathbb{C} \setminus \{te^{i\alpha}, t \geq 0\} \longrightarrow]\alpha, \alpha + 2\pi[$ defined by $A_\alpha(z) = A(r(\cos\theta + i\sin\theta)) = \theta$ is continuous.
To prove the holomorphy of the Logarithmic function, we need the following theorem

Theorem

Let Ω be an open subset of $\mathbb{C}$ and let $f : \Omega \longrightarrow \mathbb{C}$ be a holomorphic function. We assume that $f'(z) \neq 0 \; \forall z \in \Omega$, f is bijective of Ω on $f(\Omega) = \Omega'$, where Ω' is an open subset of $\mathbb{C}$.
If f^{-1}, the inverse function of f is continuous on Ω', then f^{-1} is holomorphic on Ω' and

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))}, \quad \forall\, w \in \Omega'.$$

Proof

Let $w_0 \in \Omega'$ and $z_0 = f^{-1}(w_0) \in \Omega$. For $w \in \Omega'$, there exists a unique $z \in \Omega$, such that $w = f(z)$. Since f^{-1} is continuous, if w tends to w_0, then z tends to z_0. Thus

$$\frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - f(z_0)} \underset{w \to w_0}{\longrightarrow} \frac{1}{f'(z_0)} = \frac{1}{f'(f^{-1}(w_0))}.$$

We will prove that the assumptions, f^{-1} is continuous, $f'(z) \neq 0$, $\forall z \in \Omega$ and Ω' is an open subset of $\mathbb{C}$ are hold for any holomorphic function $f : \Omega \longrightarrow \Omega'$ bijective unlike in the real case.

Corollary

The mapping $z \longmapsto \mathrm{e}^z$ is a holomorphic function, bijective of the strip $A_t = \{x + \mathrm{i}y \in \mathbb{C};\ t - \pi < y < \pi + t, x \in \mathbb{R}\}$ on $\mathbb{C} \setminus J_t$, with $J_t = \{r\mathrm{e}^{\mathrm{i}t}, r \leq 0\}$, $t \in \mathbb{R}$. It has an inverse function defined on $\mathbb{C} \setminus J_t$ with values on A_t. We denote by $\log_t$ this function.

This corollary results by theorem 2.9 and the continuity of the mapping $A(z)$.

Definition

The principal determination (branch) of the logarithm is the inverse function of the exponential function defined on $\mathbb{C} \setminus J_0$ *with values on* A_0*. We denote this function by* Log*.*

Exercise

Show that

1. $\mathrm{Log} z = \ln\sqrt{x^2+y^2} + 2\mathrm{i}\tan^{-1}\dfrac{y}{x+\sqrt{x^2+y^2}}, \quad \forall z = x+\mathrm{i}y \in \mathbb{C}\setminus J_0.$
2. $(\log_t z)' = \dfrac{1}{z}$ *for all* $t\in\mathbb{R}$ *and* $z\in\mathbb{C}\setminus J_t$.

Corollary

The function $f(z) = \log_t(z) - \log_{t'}(z)$ *is constant on each connected component of* $(\mathbb{C} \setminus J_t) \cap (\mathbb{C} \setminus J_{t'})$.

Remark

The relation $\mathrm{Log}(z_1.z_2) = \mathrm{Log} z_1 + \mathrm{Log} z_2$ *is not always valid. It suffices to take* $z_1 = \mathrm{e}^{\frac{3\mathrm{i}\pi}{4}} = z_2$, $z_1 z_2 = \mathrm{e}^{\frac{3\mathrm{i}\pi}{2}} = \mathrm{e}^{\frac{-\mathrm{i}\pi}{2}}$, $\mathrm{Log} z_1 z_2 = -\mathrm{i}\frac{\pi}{2}$ *and* $\mathrm{Log} z_1 + \mathrm{Log} z_2 = \frac{3\mathrm{i}\pi}{2} \neq -\frac{\mathrm{i}\pi}{2}$.

Proposition

For $|z| < 1$, $\mathrm{Log}(1-z) = -\sum_{n=1}^{+\infty} \frac{z^n}{n}$, *where* $\mathrm{Log}(1-z)$ *is the principal determination (branch) of the logarithmic function.*

Proof

Let $f(z) = \sum_{n=1}^{+\infty} \frac{z^n}{n}$, for $|z| < 1$, then $f'(z) = \sum_{n=0}^{+\infty} z^n = \frac{1}{1-z}$. It follows that $f(z) + \mathrm{Log}(1-z) = 0$ because $f(0) = \mathrm{Log}(1-0) = 0$.

Let Ω be an open subset of $\mathbb{C}^*$ and let $\alpha \in \mathbb{C}^*$. We define a continuous determination of z^α on Ω by any continuous function $g: \ \Omega \longrightarrow \mathbb{C}$ such that for all $z \in \Omega$, there exists a logarithm h of z such that

$$\begin{cases} g(z) = \mathrm{e}^{\alpha h(z)} \\ z = \mathrm{e}^{h(z)} \end{cases}.$$

In particular, if there exists on Ω a continuous determination (branch) of the logarithm log, then the mapping $z \longrightarrow \mathrm{e}^{\alpha \log}$ is a continuous determination of z^α on Ω. We use only this type of determination for z^α.

Definition

A function $f : \Omega \longrightarrow \mathbb{C}$ *is called analytic on* Ω *if, whenever* $z_0 \in \Omega$ *there exists a neighborhood* V *of* z_0 *and a power series*

$\sum_{n\geq 0} a_n(z - z_0)^n$ *such that* $f(z) = \sum_{n=0}^{+\infty} a_n(z - z_0)^n$ *for all* $z \in V$.

Theorem

Any analytic function is holomorphic.

Proof

This theorem results by theorem 2.1. The inverse will be proved in the chapter III. □

Theorem

If the power series $\sum_{n\geq 0} a_n z^n$ *has a radius of convergence* $R > 0$, *its sum* $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ *is analytic on the disc* $D(0, R)$.

For the proof of this theorem we need the following result on double sums.

Theorem

Let $(a_{n,m})_{n,m\in\mathbb{N}}$ be a sequence of complex numbers. If $\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}|a_{n,m}| < +\infty$, *one has*

$$\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}|a_{n,m}| = \sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}|a_{n,m}| = \sup_{N,M\in\mathbb{N}}\Big[\sum_{m=0}^{M}\sum_{n=0}^{N}|a_{n,m}|\Big],$$

and

$$\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}a_{n,m} = \sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}a_{n,m}.$$

Proof

This theorem results by Fubini's theorem. Indeed, we consider the countable measure μ on $\mathbb{N}$. If $f : \mathbb{N} \longrightarrow [0, +\infty]$ is a μ-integrable function, then $\displaystyle\int_{\mathbb{N}} f(x)\ d\mu(x) = \sum_{k=0}^{+\infty} f(k)$. Furthermore, if $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ is a bijection, then

$$\sum_{k=0}^{+\infty} f(k) = \sum_{k=0}^{+\infty} f(\sigma(k)).$$

Let $(b_{m,n})$ be a sequence of non negative real numbers. By Beppo-Levi's theorem

$$\sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} b_{m,n} = \sum_{n=0}^{+\infty} \sum_{n=0}^{+\infty} b_{m,n}.$$

It suffices to take the function f defined on $\mathbb{N} \times \mathbb{N}$ by $f(n, m) = b_{n,m}$.
In our case the function g defined on $\mathbb{N} \times \mathbb{N}$ by $g(n, m) = a_{n,m}$ is integrable, and the theorem results by Fubini's theorem.

Proof of the theorem 3.3

Let $z_0 \in D(0, R)$, $|z_0| = r_0 < R$. We shall show that
$f(z) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$, whenever $z \in D(z_0, R - r_0)$.
Let $z \in \mathbb{C}$ such that $|z - z_0| < r - r_0 < R - r_0$ and let

$$S(z) = \sum_{k=0}^{+\infty} \sum_{n=k}^{+\infty} C_n^k a_n z_0^{n-k}(z - z_0)^k \tag{7}$$

We shall prove that the series $\sum_{k=0}^{+\infty}\sum_{n=k}^{+\infty} C_n^k a_n z_0^{n-k}(z-z_0)^k$ is absolutely convergent.
Let

$$R(z) = \sum_{k=0}^{+\infty}\sum_{n=k}^{+\infty} C_n^k |a_n||z_0|^{n-k}|z-z_0|^k \qquad (8)$$

$R(z) \leq \sum_{k=0}^{+\infty}\sum_{n=k}^{+\infty} C_n^k |a_n| r_0^{n-k}(r-r_0)^k =$
$\sum_{n=0}^{+\infty} |a_n|(\sum_{k=0}^{n} C_n^k r_0^{n-k}(r-r_0)^k) = \sum_{n=0}^{+\infty} |a_n| r^n < +\infty.$ Thus the series which defines S is absolutely convergent, thus it is commutatively convergent.

$$S(z)=\sum_{k=0}^{+\infty}\frac{1}{k!}(z-z_0)^k\Big(\sum_{n=k}^{+\infty}\frac{n!}{(n-k)!}a_nz_0^{n-k}\Big),$$ this means that

$$S(z)=\sum_{k\geq 0}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$ and

$$S(z)=\sum_{n=0}^{+\infty}a_n\Big(\sum_{k=0}^{n}C_n^kz_0^{n-k}(z-z_0)^k\Big)=\sum_{n=0}^{+\infty}a_nz^n=f(z).$$

It follows that $f(z)=\sum_{k=0}^{+\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$, whenever $z\in D(z_0,R-r_0)$.

Theorem (Principle of the Analytic Continuation)

Let f be an analytic function on a domain Ω of $\mathbb{C}$ and let $z_0 \in \Omega$. The following conditions are equivalent

1. *f is identically zero on Ω.*
2. *There exists a neighborhood V of z_0 such that f is identically zero on V.*
3. *For all $n \geq 0$, $f^{(n)}(z_0) = 0$.*

Proof

We shall show that 2) $\Rightarrow$ 1) (the other properties are trivial). Let $A = \{z \in \Omega;\ f \equiv 0 \text{ on a neighborhood of z}\}$. A is a non empty open subset by hypothesis. Then it suffices to show that it is closed in Ω. Let $(z_n)_n$ be a sequence of A which converges to $a \in \Omega$. Since $z_n \in A$, then $f^{(k)}(z_n) = 0$ for all $k \in \mathbb{N}$ and by continuity $f^{(k)}(a) = 0$. f is analytic, this yields that f is zero on a neighborhood of a.

Corollary

Let f and g be two analytic functions on a domain Ω of $\mathbb{C}$. If f and g coincide on a neighborhood of a point of Ω, then they coincide on Ω.

Theorem (Principle of Isolated Zeros)

Let f be an analytic function on a domain Ω of $\mathbb{C}$. If f is not identically zero on Ω, the set of zeros of f is a discrete closed subset of Ω.

Proof

Let $A = f^{-1}\{0\}$ be the set of zeros of f. A is closed since f is continuous. Let $z_0 \in A$, by theorem 3.5, there exists k such that $f^{(k)}(z_0) \neq 0$. We choose k the smallest integer such that $f^{(k)}(z_0) \neq 0$. So the power series of f at z_0 is $f(z) = a_k(z - z_0)^k + (z - z_0)^k g(z)$, where $a_k \neq 0$ and $g(z_0) = 0$, $g(z) = \sum_{n=1}^{+\infty} a_{n+k}(z - z_0)^n$. Since $g(z_0) = 0$, there exists a neighborhood V of z_0 such that $|g(z)| < |a_k|$ for all $z \in V$. Thus $|f(z)| \geq |z - z_0|^k(|a_k| - |g(z)|) > 0$ for $z \in V \setminus \{z_0\}$. It results that z_0 is the only zero of f in V.

Corollary

If f is an analytic function and non-identically zero on a domain Ω, then any compact subset of Ω contains only a finite number of zeros of f.

Corollary

The ring of the analytic functions on a domain Ω is integral.

Corollary (The Identity Theorem)

If f and g are two analytic functions on a domain Ω, which coincides on a set admitting a cluster point (or an accumulation point) in Ω, then they coincide on Ω.