chapter (4): Point estimation.

we suppose a random variable X has a cdf $F(x|\theta)$ and pdf $f(x|\theta)$, depending on a parameter $\theta = (\theta_1, \theta_2, \cdots, \theta_m)$.
Using data $x_1, \cdots, x_n$, we want to find an estimate of θ.

① Method of moments:

we solve the equation:

$$E(X^k) = \frac{\sum_{i=1}^{n} (X_i)^k}{n}, \quad \text{for } k = 1, \cdots, m.$$

Ex: let $X \sim Exp(\lambda)$. Find an estimate of λ using the method of moments. Data $X_1, \cdots, X_n$ is given.

$$E(X) = \frac{\sum_{i=1}^{n} X_i}{n} = \bar{X}.$$

$$\frac{1}{\lambda} = \bar{X} \Leftrightarrow \lambda = \frac{1}{\bar{X}}.$$

$$\hat{\lambda} = \frac{1}{\bar{X}}.$$

(Data: 2, 3, 10, 5 $\longrightarrow$ $\hat{\lambda} = \frac{1}{5} = 0.2$).

Ex: let $X \sim Gamma(\alpha, \beta)$. Find an estimate of α and β using method of moments.

$$\begin{cases} E(X) = \bar{X} \\ E(X^2) = \dfrac{\sum_{i=1}^{n} x_i^2}{n} = \bar{Y} \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha/\beta = \bar{X} \Rightarrow \alpha = \beta \bar{X} \\ \dfrac{\alpha}{\beta^2} + \dfrac{\alpha^2}{\beta^2} = \bar{Y} \Rightarrow \dfrac{\bar{X}}{\beta} + \bar{X}^2 = \bar{Y} \end{cases}$$

$$\Rightarrow \hat{\beta} = \frac{\bar{X}}{\bar{Y} - \bar{X}^2} = \frac{n}{n-1} \frac{\bar{X}}{S^2} \quad \left(S^2 = \frac{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}{n-1} \right).$$

$$\hat{\alpha} = \frac{\bar{X}^2}{\bar{Y} - \bar{X}^2} = \frac{n}{n-1} \frac{\bar{X}^2}{S^2}.$$

(2) Maximum likelihood method:

we define the likelihood function:

$$L(\theta) = \prod_{i=1}^{n} f(x_i | \theta).$$

The maximum likelihood estimate (MLE) of θ is the value maximizing the function $L(\theta)$.

Ex: Let $X \sim Exp(\lambda)$. Find the maximum likelihood estimate of λ. $(X_1, \dots, X_n)$.

$$L(\lambda) = \lambda e^{-\lambda x_1} . \lambda e^{-\lambda x_2} \dots \lambda e^{-\lambda x_n}$$

$$= \lambda^n e^{-n\lambda \bar{X}}.$$

$$\ell(\lambda) = \ln(L(\lambda)) = n \ln(\lambda) - n\lambda \bar{X}.$$

$$\ell'(\lambda) = \frac{n}{\lambda} - n\bar{X} = 0 \Leftrightarrow \hat{\lambda} = \frac{1}{\bar{X}}.$$

...

Ex: let $X \sim N(\mu, \sigma^2)$. Find the maximum likelihood estimate of μ and σ^2.

- $$L(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_1-\mu)^2}{2\sigma^2}} \cdots \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_n-\mu)^2}{2\sigma^2}}$$

$$= (2\pi\sigma^2)^{-n/2} e^{-\sum_{i=1}^{n} \frac{(x_i-\mu)^2}{2\sigma^2}}.$$

- $$\ell(\mu, \sigma^2) = -\frac{n}{2}\left(\ln(2\pi) + \ln(\sigma^2)\right) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i-\mu)^2$$

- $$\frac{\partial \ell(\mu, \sigma^2)}{\partial \mu} = + \frac{1}{\cancel{2}\sigma^2} \sum_{i=1}^{n} \cancel{2}(x_i-\mu) = 0.$$

$$\frac{\partial \ell(\mu, \sigma^2)}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^{n} (x_i-\mu)^2 = 0.$$

$$\Rightarrow \quad \cancel{\hat{\mu} = \bar{x}} \quad \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \mu = 0$$

$$\sum x_i = n\mu \Rightarrow \mu = \frac{1}{n} \sum x_i = \bar{x}.$$

$$\boxed{\hat{\mu} = \bar{X}}$$

$$\frac{n}{\cancel{2}\sigma^2} = \frac{1}{\cancel{2}\sigma^4} \sum_{i=1}^{n} (X_i - \bar{X})^2.$$

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2 = \frac{n-1}{n} S^2.$$

(3) Maximum likelihood method for Grouped data:

let:

Value	# of observations
(c_0, c_1)	m_1
(c_1, c_2)	m_2
$\vdots$	$\vdots$
(c_{n-1}, c_n)	m_n

The likelihood function:

$$L(\theta) = \prod_{i=1}^{n} \left(F(c_i) - F(c_{i-1}) \right)^{m_i}.$$

Example: $X \sim Exp(\lambda)$.

x	# of observations
$(0, 10)$	2
$(12, 20)$	1
$(20, \infty)$	3
Total	6

Find the MLE of λ?

$$F(x) = 1 - e^{-\lambda x}.$$

$$L(\lambda) = (F(10) - F(0))^2 (F(20) - F(12))^1 (F(\infty) - F(20))^3.$$

$$= (1 - e^{-10\lambda})^2 (e^{-12\lambda} - e^{-20\lambda}) (e^{-20\lambda})^3.$$

$$\ell(\lambda) = 2 \ln(1 - e^{-10\lambda}) + \ln(e^{-12\lambda} - e^{-20\lambda}) - 60\lambda.$$

$$\frac{\partial}{\partial \lambda} \ell(\lambda) = \frac{(2)(10) e^{-10\lambda}}{1 - e^{-10\lambda}} + \frac{-12 e^{-12\lambda} + 20 e^{-20\lambda}}{e^{-12\lambda} - e^{-20\lambda}} - 60 = 0.$$

$$\hat{\lambda} = 0.0352.$$

(4)

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④ ML method for mixed data:

value	# of observations
(c_0, c_1)	m_1
$\vdots$	$\vdots$
(c_{n-1}, c_n)	m_n
x_1	k_1
$\vdots$	$\vdots$
x_p	k_p

$$L(\theta) = \left(\prod_{j=1}^{p} (f(x_j))^{k_j}\right)\left(\prod_{j=1}^{n} (F(c_j) - F(c_{j-1}))^{m_j}\right)$$

Example: let $S_X(x) = \frac{\theta^4}{(\theta^2 + x^2)^2}$, $x \geq 0$.

Two values of X are observed to be 2 and 4, and the third value exceeds 4. Calculate the MLE of θ.

$$f_X(x) = \frac{\theta^4 \, 2x \, 2(\theta^2+x^2)}{(\theta^2+x^2)^4} = \frac{4x\theta^4}{(\theta^2+x^2)^3}.$$

$$L(\theta) = f(2)f(4)(F(\infty) - F(4)).$$

$$= \frac{8\theta^4}{(\theta^2+4)^3} \cdot \frac{16\theta^4}{(\theta^2+16)^3} \cdot \frac{\theta^4}{(\theta^2+16)^2}.$$

$$= 128 \frac{\theta^{12}}{(\theta^2+4)^3(\theta^2+16)^5}.$$

$$\ell(\theta) = \ln(128) + 12\ln(\theta) - 3\ln(\theta^2+4) - 5\ln(\theta^2+16).$$

$$\frac{\partial}{\partial\theta}\ell(\theta) = \frac{12}{\theta} - \frac{6\theta}{\theta^2+4} - \frac{10\theta}{\theta^2+16} = 0.$$

$$= \frac{-4\theta^4 + 104\theta^2 + 768}{\theta(\theta^2+4)(\theta^2+16)} = 0$$

$\theta^2 = 32$, $\hat{\theta} = \sqrt{32} = 5.66$.

⑤

(5) ML method for incomplete data:

(5.1) * Censored data:

Left Censored: An observation less than a value d is recorded as d.

Rigth Censored: An observation greater than a value u, is recorded as u.

* The MLE for Censored data is done in the same way as for group data.

Exaple: A ground-up loss variable X has a policy limit of 30. Let 6 payments:

$Y^L \leftarrow$ 20, 25, 27, 28, 30, 30.

let $X \sim Exp(\lambda)$. $\hat{\lambda} = ?$

$$F_{Y^L}(y) = \begin{cases} 0 & y<0 \\ F_X(y) & 0 \le y < u \\ 1 & y \ge u \end{cases}$$

$$f_{Y^L}(y) = \begin{cases} 0 & y<0,\ y>u \\ f_X(y) & 0 \le y < u. \\ 1-F_X(u) & y=u \end{cases}$$

$f = f_{Y^L}$.
$S = S_{Y^L}$

$$L(\lambda) = f(20)f(25)f(27)f(28)\left(S(30)\right)^2.$$
$$= \lambda e^{-20\lambda}\lambda e^{-25\lambda}\lambda e^{-27\lambda}\lambda e^{-28\lambda}\left(e^{-30\lambda}\right)^2.$$
$$= \lambda^4 e^{-160\lambda}.$$

$$\ell(\lambda) = 4\ln(\lambda) - 160\lambda$$

$$\frac{\partial \ell(\lambda)}{\partial \lambda} = \frac{4}{\lambda} - 160 = 0 \rightarrow \lambda = \frac{4}{160} = 0.035.$$

(6)

(5.2) Truncated data:

$X: \quad x_1, x_2, \ldots, x_n$.

but some ~~obje~~ observations are deleted: $y_1, \ldots, y_k$.

let $B = \{$ deleted observations $\}$.

$$L(\theta) = \prod_{i=1}^{k} f_{X|B^c}(y_i).$$

Example:

Ex: A ground up loss X has a deductible of 7 applied. A random sample of 6 insurance payments (after deductible is applied) is given.

3, 6, 7, 8, 10, 12.

Suppose $X \sim Exp(\theta)$. Find the MLE of θ?

$$L(\theta) = \prod_{i=1}^{6} f_{Y^P}(x_i) = \prod_{i=1}^{6} \frac{f_X(x_i+7)}{S_X(7)}$$

$$= \prod_{i=1}^{6} \frac{\theta e^{-\theta(x_i+7)}}{e^{-7\theta}} = \theta^6 e^{-\theta \sum x_i}$$

$$= \theta^6 e^{-46\theta}$$

$$\ell(\theta) = 6\ln(\theta) - 46\theta$$

$$\frac{\partial}{\partial \theta}\ell(\theta) = \frac{6}{\theta} - 46 = 0 \iff \theta = \frac{6}{46} = 0.13$$

Ex: For a dental policy, you are given:

* Ground-up losses $X \sim Exp(\theta)$.
* Losses under 50 are not reported to the insurer.
* For each loss over 50, there are a deductible of 50 and a policy limit of 400.
* A random sample: 50, 150, 200, 350^+, 350^+

where + indicates that the original loss exceeds 400.

Find the MLE of θ?

$$Y^L = (X \wedge 400) - (X \wedge 50) = \begin{cases} 0 & X < 50 \\ X-50 & 50 < X < 400 \\ 350 & X > 400 \end{cases}$$

$$L(\theta) = \prod_{i=1}^{5} f_{Y^P}(x_i) = f_{Y^P}(50)\, f_{Y^P}(150)\, f_{Y^P}(200) \left(f_{Y^P}(350)\right)^2$$

$$= \frac{f_X(100)}{S_X(50)} \frac{f_X(200)}{S_X(50)} \frac{f_X(250)}{S_X(50)} \left(\frac{S_X(400)}{S_X(50)}\right)^2$$

8

$$L(\theta) = \frac{\theta e^{-100\theta}\ \theta e^{-200\theta}\ \theta e^{-250\theta}}{(e^{-500\theta})^3}\left(\frac{e^{-400\theta}}{e^{-500\theta}}\right)^2$$

$$= \theta^3 e^{-1100\theta}$$

$$\ell(\theta) = 3\ln(\theta) - 1100\theta \rightarrow \frac{\partial}{\partial\theta} = \frac{3}{\theta} - 1100 \rightarrow \theta = \frac{3}{1100} = 0.0027$$

⑥ Method of percentiles :

- Percentile estimate : $\hat{\pi}_g$, $g \in (0,1)$.
  - * order the data .
  - * find p such that :

$$\frac{p}{n+1} \le g < \frac{p+1}{n+1}. \qquad p \le g(n+1) \le p+1$$

  - * $$\hat{\pi}_g = (n+1)\left[\left(\frac{p+1}{n+1} - g\right)x_{(p)} + \left(g - \frac{p}{n+1}\right)x_{(p+1)}\right]$$

Ex: Find the percentile $\hat{\pi}_{0.4}$:

1, 2, 5, 10 | 12, 17, 20, 21, 25, 30 .

$n = 10$, $g(n+1) = (10+1)(0.4) = 4.4$.

$p=4 \qquad \frac{4}{11} \le 0.4 \le \frac{5}{11}$.

$$\hat{\pi}_{0.4} = 11\left(\left(\frac{5}{11} - 0.4\right)(10) + \left(0.4 - \frac{4}{11}\right)(12)\right)$$
$$= 10.8$$

- Percentile method :

$f(x|\theta)$; $\underset{\sim}{\theta} = (\theta_1, \dots, \theta_k)$.

we fix k percentiles $\pi_{g_1}, \dots, \pi_{g_k}$ and solve the equations :

$$\pi_{g_i} = \hat{\pi}_{g_i} \quad \text{for } i = 1, \dots, k.$$

Ex : Let the Burr distribution:

$$F(x|\theta,\gamma) = 1 - \left(\frac{1}{1+(x/\theta)^{\gamma}}\right)^{2}.$$

* 195, 200, | 270, 280, | 300, 360

* Use empirical estimate of the 30th and 65th percentiles to estimate θ and γ.

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* $\hat{\pi}_{0.3}$: $2 \le (0.3)7 = 2.1 \le 3$.

$$\frac{2}{7} \le 0.3 < \frac{3}{7}.$$

$$\hat{\pi}_{0.3} = 7\left(\left(\frac{3}{7}-0.3\right)200 + \left(0.3-\frac{2}{7}\right)270\right).$$

$$= 207$$

* $\hat{\pi}_{0.65}$: $4 \le 0.65 \times 7 = 4.55 < 5$

$$\frac{4}{7} \le 0.65 \le \frac{5}{7}$$

$$\hat{\pi}_{0.65} = 7\left(\left(\frac{5}{7}-0.65\right)280 - \left(0.65-\frac{4}{7}\right)300\right)$$

$$= 291.$$

* $\begin{cases} \pi_{0.3} = \hat{\pi}_{0.3} \\ \pi_{0.65} = \hat{\pi}_{0.65} \end{cases} \Leftrightarrow \begin{cases} F(\pi_{0.3}) = 0.3 = F(\hat{\pi}_{0.3}) \\ F(\pi_{0.65}) = 0.65 = F(\hat{\pi}_{0.65}). \end{cases}$

$a = \hat{\pi}_{0.3}$

$b = \hat{\pi}_{0.65}$

$$\Leftrightarrow \begin{cases} 1 - \left(\frac{1}{1+\left(\frac{a}{\theta}\right)^{\gamma}}\right)^{2} = 0.3 \\ 1 - \left(\frac{1}{1+\left(\frac{b}{\theta}\right)^{\gamma}}\right)^{2} = 0.65 \end{cases}$$

$$\Leftrightarrow \begin{cases} \left(1+\left(\frac{a}{\theta}\right)^{\gamma}\right)^{2} = \frac{1}{0.7} & \left(\frac{a}{\theta}\right)^{\gamma} = \sqrt{\frac{1}{0.7}} - 1 = \\ \left(1+\left(\frac{b}{\theta}\right)^{\gamma}\right)^{2} = \frac{1}{0.35} & \left(\frac{b}{\theta}\right)^{\gamma} = \sqrt{\frac{1}{0.35}} - 1 = \end{cases}$$

$$\Rightarrow \left(\frac{a}{b}\right)^{\gamma} = 0.2828 \Rightarrow \gamma = \frac{\ln(0.2828)}{\ln(207/291)} = 3.7$$

$$\Rightarrow \theta = a\left(\sqrt{\frac{1}{0.7}} - 1\right)^{-1/\gamma} = 321.9.$$

7 Fisher information:

let a random sample $X_1, \dots, X_n$. (i.i.d).

with the pdf $f(x|\theta)$.

we define the Fisher information on θ:

$$I(\theta) = E\left(\ell'(\theta)\right)^2 = -E\,\ell''(\theta),$$

where $\ell(\theta) = \ln(L(\theta))$.

Ex: $X_1, \dots, X_n \sim \text{Poisson}(\theta)$.

Compute $I(\theta)$?

$$L(\theta) = \bar{e}^{\theta} \frac{\theta^{x_1}}{x_1!} \cdots \bar{e}^{\theta} \frac{\theta^{x_1}}{x_1!}.$$

$$= \bar{e}^{n\theta} \frac{\theta^{n\bar{x}}}{\prod_{i=1} x_i!}.$$

$$\ell(\theta) = -n\theta + n\bar{x}\ln(\theta) - \ln(-).$$

$$\ell'(\theta) = -n + \frac{n\bar{X}}{\theta}.$$

$$\ell''(\theta) = -\frac{n\bar{X}}{\theta^2}.$$

$$I(\theta) = E\,\frac{n\bar{X}}{\theta^2} = \frac{n}{\theta^2} E(\bar{X}) = \frac{n}{\theta^2}\theta = \frac{n}{\theta}.$$

Theorem: let $\hat{\theta}$ the MLE of θ. Then

$$\sqrt{I(\theta)}\,(\hat{\theta} - \theta) \sim N(0,1).$$

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