Chapter (3) : Modifications of loss distributions.

insurance policy $\longrightarrow$ risk X

$\downarrow$

payment Y

in some cases, Y is different from X.

(1) Ordinary policy deductible :

Risk X $\longrightarrow$ $X > d$

$\downarrow$

$Y^L = X - d$.

$d = 100$; $\longrightarrow X = 120 \longrightarrow Y^L = 20$

$\ \longrightarrow X = 80 \longrightarrow Y^L = 0$.

$X \leq d \longrightarrow Y^L = 0$.

Y^L called Cost-per loss is given :

$$Y^L = (X - d)_+ .$$

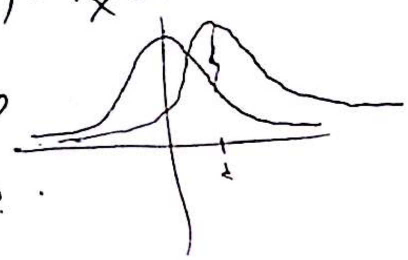
Y^L is called also left censored and shifted variable.

- CDF of Y^L: -

$$F_{Y^L}(y) = P(Y^L \le y)$$
$$= P((X-d)_+ \le y)$$
$$= P((X-d)_+ \le y;\ X \le d) + P((X-d)_+ \le y,\ X>d)$$
$$= P(0 \le y;\ X \le d) + P(X-d \le y;\ X>d)$$

$y \geq 0$:

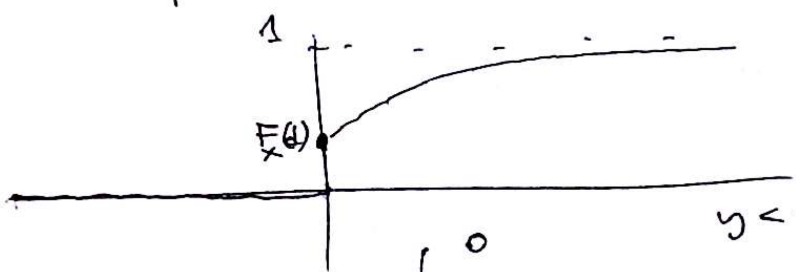
$$F_{Y^L}(y) = F_X(d) + F_X(y+d) - F_X(d)$$



$$F_{Y^L}(y) = \begin{cases} F_X(y+d) & ; y \geq 0 \\ 0 & ; y<0 \end{cases}$$

Remark: • $F_{Y^L}(0) = F_X(d)$, $F_{Y^L}(0^-) = 0$.

- Y^L is a mixed distribution.



- $$f_{Y^L}(y) = \begin{cases} 0 & y<0 \\ F_X(d) & y=0 \\ f_X(y+d) & y>0 \end{cases}$$

- mgf of Y^L: (X continuous)

$$M_{Y^L}(t) = E\, e^{tY} = E\, e^{t(X-d)_+}$$
$$= \int e^{t(x-d)_+} f_X(x)\,dx = \int_0^d f_X(x)\,dx + \int_d^\infty e^{t(x-d)} f_X(x)\,dx.$$
$$= F_X(d) + e^{-td}\left(\int_d^\infty e^{tx} f_X(x)\,dx\right)$$
$$= F_X(d) + e^{-td}\left(M_X(t) - \int_0^d e^{tx} f_X(x)\,dx\right).$$

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Example: Determine the pdf, cdf and sdf for Y^L if the ground-up loss amount function has an exponential distribution with mean $\frac{1}{\theta}$ and an ordinary deductible of d.

$X \sim Exp(\theta)$; $f_X(x) = \theta e^{-\theta x}$; $x \geq 0$,

$$F_X(x) = \begin{cases} 1 - e^{-\theta x} & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$$

$$S_X(x) = \begin{cases} e^{-\theta x} & ; x \geq 0 \\ 1 & ; x < 0 \end{cases}$$

- ~~$F_{Y^L}(y) = F_X(y+d) = 1 - e^{-\theta(y+d)}$;~~

$$F_{Y^L}(y) = \begin{cases} 0 & y < 0 \\ F_X(y+d) & y \geq 0 \end{cases}$$

$$= \begin{cases} 0 & y < 0 \\ 1 - e^{-\theta(y+d)} & y \geq 0 \end{cases}$$

$$f_{Y^L}(y) = \begin{cases} 0 & y < 0 \\ 1 - e^{-\theta d} & y = 0 \\ \theta e^{-\theta(y+d)} & y > 0 \end{cases}$$

$$S_{Y^L}(y) = \begin{cases} 1 & y < 0 \\ e^{-\theta(y+d)} & y \geq 0 \end{cases}$$

- $E(Y_L) = E(X-d)_+ = \int_0^\infty (x-d)_+ f_X(x)\,dx$

$$= \int_d^\infty \underbrace{(x-d)}_{u} \underbrace{f_X(x)\,dx}_{dv} = \int_d^\infty \underbrace{(x-d)}\,\underbrace{\theta e^{-\theta x}\,dx}$$

$$= -(x-d)e^{-\theta x}\Big|_d^\infty + \int_d^\infty e^{-\theta x}\,dx = \frac{-1}{\theta} e^{-\theta x}\Big|_d^\infty = \frac{1}{\theta} e^{-\theta d}.$$

$$\bullet\ m_{Y^L}(t) = E e^{tY^L} = E e^{t(X-d)_+}$$

$$= \int_0^\infty e^{t(x-d)_+} \theta e^{-\theta x} dx = \int_0^d \theta e^{-\theta x} dx + \int_d^\infty e^{t(x-d)} e^{-\theta x} \theta \, dx$$

$$= 1 - e^{-\theta d} + e^{-dt} \int_d^\infty \theta e^{(t-\theta)x} dx$$

$$= 1 - e^{-\theta d} + e^{-dt} \frac{\theta}{t-\theta} e^{(t-\theta)x} \Big|_d^\infty$$

$t<\theta$

$$= 1 - e^{-\theta d} + e^{-dt} \frac{\theta}{t-\theta} \left(-e^{(t-\theta)d}\right)$$

$$= \underbrace{1 - e^{-\theta d}}_{1-q} + \frac{\theta}{\theta - t} \underbrace{e^{-\theta d}}_{q}$$

- Now we define the cost-per payment:

$$Y^P = (Y^L \mid X > d) = (Y^L \mid Y^L > 0)$$

A random variable X, event $B \doteq (X > c)$

$$Y = (X \mid X > c).$$

$$F_Y(y) = P(Y \le y \mid X > c) = P(X \le y \mid X > c) = \frac{P(c < X \le y)}{P(X > c)}$$

$$= \begin{cases} 0 & y < c \\ \dfrac{F_X(y) - F_X(c)}{1 - F_X(c)} & y \ge c \end{cases}$$

$$f_Y(y) = \begin{cases} 0 & y < c \\ \dfrac{f_X(y)}{1 - F_X(c)} & y \ge c \end{cases}$$

$$E(Y) = \int_c^\infty \frac{y f_X(y)}{S_X(c)} dx =$$

- CDF of Y^l:

$$F_{Y^l}(y) = \begin{cases} 0 & y < 0 \\ \dfrac{F_{Y^L}(y) - F_{Y^L}(0)}{1 - F_{Y^L}(0)} & y \geq 0 \end{cases}$$

$$= \begin{cases} 0 & y < 0 \\ \dfrac{F_X(y+d) - F_X(d)}{1 - F_X(d)} & y \geq 0 \end{cases}$$

- f_{Y^l}:

$$f_{Y^l}(y) = \begin{cases} 0 & y < 0 \\ \dfrac{f_X(y+d)}{1 - F_X(d)} & y \geq 0 \end{cases}$$

- Mean of Y^l:

$$E(Y^l) = \int_0^\infty y \, \frac{f_X(y+d)}{1 - F_X(d)} \, dy$$

$$E(Y^L) = \int_0^\infty y \, dF_{Y^L}(y) = 0 \cdot f_{Y^L}(0) + \int_0^\infty y \cdot f_X(y+d) \, dy.$$

$$E(Y^l) = \frac{E(Y^L)}{1 - F_X(d)}.$$

- $E(X \wedge d) = ?$

$$Y^L = X - X \wedge d = \begin{cases} 0 & X < d \\ X - d & X > d \end{cases}$$

$$E Y^L = E(X) - \boxed{E(X \wedge d)}$$

(2) Franchise deductible:

- risk $X \longrightarrow X > d \longrightarrow Y^L = X$

  $X \le d \longrightarrow Y^L = 0$

  $$Y^L = \begin{cases} 0 & ; X \le d \\ X & ; X > d. \end{cases}$$

- $Y^P = (Y^L \mid Y^L > 0) = (X \mid X > d).$

- CDF of Y^L: $F_{Y^L}(y) = P(Y^L \le y)$

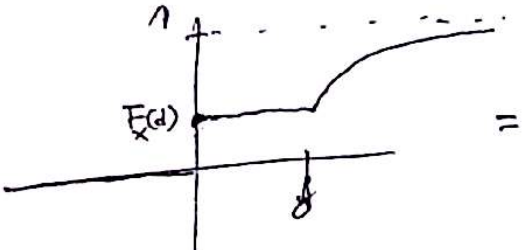
$$= P(Y^L \le y \,;\, X \le d) + P(Y^L \le y \,;\, X > d)$$

$$= P(0 \le y \,;\, X \le d) + P(X \le y \,;\, X > d).$$

$\underline{y \ge 0}$:

$$= F_X(d) + \begin{cases} 0 & y < d \\ F_X(y) - F_X(d) & y \ge d \end{cases}$$

$$= \begin{cases} F_X(d) & 0 \le y < d \\ F_X(y) & y \ge d \\ 0 & y < 0 \end{cases}$$



- $$f_{Y^L}(y) = \begin{cases} 0 & y < 0 \\ F_X(d) & y = 0 \\ 0 & 0 \le y < d \\ f_X(y) & y \ge d \end{cases}$$

- CDF of Y^P:

$$F_{Y^P}(y) = \begin{cases} 0 & y<0 \\ \dfrac{F_{Y^L}(y) - F_{Y^L}(0)}{1 - F_{Y^L}(0)} ; & y \geq 0. \end{cases}$$

$$= \begin{cases} 0 & y<0, \\ 0 & 0 \leq y < d \\ \dfrac{F_X(y) - F_X(d)}{1 - F_X(d)} & y \geq d \end{cases}$$

$$\therefore \; = \begin{cases} 0 & y<d \\ \dfrac{F_X(y) - F_X(d)}{1 - F_X(d)} & y \geq d ; \end{cases}$$

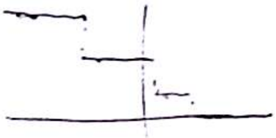
- $$f_{Y^P}(y) = \begin{cases} 0 & y<d. \\ \dfrac{f_X(y)}{1 - F_X(d)} & y \geq d. \end{cases}$$

Example: $X \sim Exp(\theta)$. Insurance policies are subject to a franchise deductible of d. Compute the cdf, pdf, sdf, hazard rate functi of Y^L and Y^P.

- Y^L:

$$F_{Y^L}(y) = \begin{cases} 0 & y<0. \\ 1 - e^{-\theta d} & 0 \leq y < d \\ 1 - e^{-\theta y} & y \geq d \end{cases}$$

$$f_{Y^L}(y) = \begin{cases} 0 & y<d, \; y \neq 0 \\ 1 - e^{-\theta d} & y=0 \\ \theta e^{-\theta y} & y \geq d \end{cases}$$



$$S_{Y^L}(y) = \begin{cases} 1 & y<0 \\ e^{-\theta d} & 0 \le y < d \\ e^{-\theta y} & y \ge d . \end{cases}$$

$$h_{Y^L}(y) = \begin{cases} 0 & y<0 \\ \dfrac{1-e^{-\theta d}}{e^{-\theta d}} & y=0 \\ 0 & 0<y<d \\ \theta & y \ge d \end{cases}$$

- Y^P:

$$F_{Y^P}(y) = \begin{cases} 0 & y<d \\ \dfrac{e^{-\theta d} - e^{-\theta y}}{e^{-\theta d}} = 1-e^{-\theta(y-d)} & y \ge d \end{cases}$$

$y \le d$,

$$f_{Y^P}(y) = \begin{cases} 0 \\ \theta\, e^{-\theta(y-d)} & y \ge d \\ & y<d \end{cases}$$

$$S_{Y^P}(y) = \begin{cases} 1 \\ e^{-\theta(y-d)} & y \ge d \end{cases}$$

$$h_{Y^P}(y) = \begin{cases} 0 & y<d \\ \theta & y \ge d \end{cases}$$

- Mean of Y^L and Y^P:

$$Y^L = X - X\wedge d + d\,1_{(X>d)} = \begin{cases} 0 & X \le d \\ X & X>d . \end{cases}$$

$$E(Y^L) = E(X) - E(X\wedge d) + d\, S_X(d) .$$

$$E(Y^P) = \frac{E(Y^L)}{1-F_X(d)} .$$

(8)

Ex: Let $X \sim Exp(\theta)$ with franchise deductible of d.

Compute $E(Y^L)$ and $E(Y^P)$.

- $$\boxed{E(X \wedge d) = \int_0^d S_X(x)\,dx}$$

$$E(Y^L) = \int_{-\infty}^{0^-} + \int_{0^-}^{0} + \int_0^d + \int_d^\infty$$

$$= 0 + 0 + 0 + \int_d^\infty y\theta e^{-\theta y}\,dy.$$

$$E(X \wedge d) = \int_0^d e^{-\theta x}\,dx = -\frac{1}{\theta} e^{-\theta x}\Big|_0^d$$

$$= \frac{1}{\theta}\left(1 - e^{-\theta d}\right).$$

$$E(Y^L) = E(X) - E(X \wedge d) + d\,S_X(d)$$

$$= \frac{1}{\theta} - \frac{1}{\theta}\left(1 - e^{-\theta d}\right) + d e^{-\theta d} = \left(\frac{1}{\theta} + d\right) e^{-\theta d}.$$

$$E(Y^P) = \frac{E(Y^L)}{S_X(d)} = \frac{\left(\frac{1}{\theta} + d\right) e^{-\theta d}}{e^{-\theta d}}$$

$$= \frac{1}{\theta} + d.$$

3 Policy limit:

$$\begin{array}{ccccc} \text{risk } X & \longrightarrow & X < u & \longrightarrow & Y = X \\ \downarrow & & & & \\ X \geq u & \longrightarrow & Y = u & & \end{array}$$

The limited loss random variable

$$Y = X \wedge u .$$

- CDF of Y:

$$F_Y(y) = P(Y \leq y) = P(X \wedge u \leq y)$$

$$= P(X \wedge u \leq y;\ X \leq u) + P(X \wedge u \leq y;\ X > u)$$

$$= P(X \leq y,\ X \leq u) + P(u \leq y;\ X > u).$$

$$= F_X(y \wedge u) + \begin{cases} 0 & u > y \\ 1 - F_X(u) & u \leq y \end{cases}$$

$$= \begin{cases} F_X(y) & y < u \\ 1 & y \geq u \end{cases}$$

- PDF of Y:

$$f_Y(y) = \begin{cases} f_X(y) & y < u \\ 1 - F_X(u) & y = u \\ 0 & y > u \end{cases}$$

- $$E(Y) = E(X \wedge u) = \int_0^u S_X(x)\,dx .$$

Example: let $X \sim \text{Gamma}(1, \beta)$.

- $$F_Y(y) = \begin{cases} 0 & y < 0 \\ 1 - e^{-\beta y} & 0 \leq y < u \\ 1 & y \geq u . \end{cases}$$

- $f_Y(y) = \begin{cases} 0 & y<0 \\ \beta e^{-\beta y} & 0 \le y < u \\ e^{-\beta u} & y = u \\ 0 & y > u \end{cases}$

- $E(Y) = \int_0^u e^{-\beta x} dx = -\frac{1}{\beta} e^{-\beta x} \Big|_0^u$

$= \frac{1}{\beta}(1 - e^{-\beta u})$.

(4) Coinsurance :

An insurance policy is defined with a coinsurance factor $0<\alpha<1$.

risk $X \longrightarrow Y = \alpha X$.

- $F_Y(y) = F_X\left(\frac{y}{\alpha}\right)$
- $f_Y(y) = \frac{1}{\alpha} f_X\left(\frac{y}{\alpha}\right)$.

(5) Inflation effect :

We suppose that r is the inflation coefficient. Then a payment Y becomes $(1+r)Y$.

Suppose now an ordinary policy deductible with risk X.

$$X \longrightarrow (1+r)X.$$

$$\downarrow$$

$$Y^L = ((1+r)X - d)_+$$
$$= (1+r)\left(X - \frac{d}{1+r}\right)_+$$

(6) Combination:

we suppose we have an ordinary deductible of d, limit of u, with coinsurance factor α and inflation coefficient r.

For a risk X, we have:

$$Y^L = \begin{cases} 0 & X < \frac{d}{1+r} \\ \alpha((1+r)X - d) & \frac{d}{1+r} \le X < \frac{u}{1+r} \\ \alpha(u-d) & X \geq \frac{u}{1+r} \end{cases}$$

we write $Y^L = \alpha\left[(1+r)X \wedge u - (1+r)X \wedge d\right]$.

Ex: $r = 0.01$; $\alpha = 0.8$; $d = 200$; $u = 500$;

$x = 100 \xrightarrow{x=101} Y^L = 0$.

$x = 300 \xrightarrow{x=303} Y^L = 82.4$.

$x = 600 \xrightarrow{x=606} Y^L = 240$.

- CDF of Y^L:

  * $\alpha = 1$; $r = 0$.

$$F_{Y^L}(y) = P(X \wedge u - X \wedge d \le y)$$

$$= P\left(X \wedge u - X \wedge d \le y;\ \begin{matrix} X < d \\ d \le X < u \\ X \geq u \end{matrix}\right)$$

$$= P(0 \le y;\ X < d) + P(X \le d + y;\ d \le X \le u) + P(u - d \le y;\ X \geq u).$$

$$= F_X(d) + F_X((d+y) \wedge u) - F_X(d) + \begin{cases} 0, & y < u-d \\ 1 - F_X(u), & \end{cases}$$

$$= \begin{cases} F_X(d+y) & y < u-d \\ 1 & y \geq u-d \end{cases}$$

In general:

$$F_{Y^L}(y) = \begin{cases} F_X\left(\dfrac{d+y/\alpha}{1+r}\right) & ;\ 0 \le y < \alpha(u-d) \\ 1 & ;\ y \ge \alpha(u-d). \\ 0 & y < 0 \end{cases}$$

- $Y^\ell = Y^L \mid Y^L > 0$:

$$F_{Y^\ell}(y) = \begin{cases} 0 & y < 0 \\ \dfrac{F_{Y^L}(y) - F_{Y^L}(0)}{1 - F_{Y^L}(0)} & y \ge 0 \end{cases}$$

$$= \begin{cases} 0 & y < 0 \\ \dfrac{F_X\left(\frac{d+y/\alpha}{1+r}\right) - F_X\left(\frac{d}{1+r}\right)}{1 - F_X\left(\frac{d}{1+r}\right)} & y < \alpha(u-d) \\ 1 & y \ge \alpha(u-d) \end{cases}$$

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$$f_{Y^L}(y) = \begin{cases} 0 & y < 0 \\ F_X\left(\dfrac{d}{1+r}\right) & y = 0 \\ \dfrac{1}{\alpha(1-r)} f_X\left(\dfrac{d+y/\alpha}{1+r}\right) & 0 < y < \alpha(u-d) \\ 1 - F_X\left(\dfrac{u}{1+r}\right) & y = \alpha(u-d) \\ 0 & y > \alpha(u-d) \end{cases}$$

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$$f_{Y^\ell}(y) = \begin{cases} 0 & y < 0,\ y > \alpha(u-d) \\ \dfrac{\frac{1}{\alpha(1-r)} f_X\left(\frac{d+y/\alpha}{1+r}\right)}{1 - F_X\left(\frac{d}{1+r}\right)} & \cancel{y=0}\quad 0 \le y \le \alpha(u-d) \end{cases}$$

Ex: A health insurance has a deductible of 200, a policy limit of 5,000, and a coinsurance factor of 80%. Calculate the expected claim amount per loss and the expected claim amount per payment if loss follow an exponential distribution with mean 1,000.

- $Y^L = \alpha(X\wedge u - X\wedge d)$.

$$E(X\wedge u) = \int_0^\infty (x\wedge u)\lambda e^{-\lambda x}\,dx$$

$$= \int_0^u x\lambda e^{-\lambda x}\,dx + \int_u^\infty u\lambda e^{-\lambda x}\,dx$$

$$= -xe^{-\lambda x}\Big|_0^u + \int_0^u e^{-\lambda x}\,dx + u\,S_X(u)$$

$$= -\cancel{ue^{-\lambda u}} + 0 + \frac{1}{\lambda}(1-e^{-\lambda u}) + \cancel{ue^{-\lambda u}} = \frac{1}{\lambda}(1-e^{-\lambda u}) + 0.$$

$$= \cancel{-\lambda u e^{-\lambda u} + \frac{1}{\lambda}}$$

$$= \cancel{\left(-\lambda u - \frac{1}{\lambda}\right)e^{-\lambda u} + \frac{1}{\lambda} + u}$$

$$EY^L = \alpha\left(E(X\wedge u) - E(X\wedge d)\right)$$

$$= \alpha\left(0 + \frac{1}{\lambda}(1-e^{-\lambda u}) - 0 - \frac{1}{\lambda}(1-e^{-\lambda d})\right)$$

$$= \frac{\alpha}{\lambda}\left(e^{-\lambda d} - e^{-\lambda u}\right).$$

$\lambda = 0.001$

$u = 5{,}000$

$d = 200$, $\alpha = 0.8$

$\therefore EY^L = 649.59$.

- $Y^P = Y^L \mid Y^L > 0$. $\quad EY^P = \dfrac{EY^L}{S_{Y^L}(0)} = \dfrac{EY^L}{S_X(d)}$

$$= \frac{649.59}{e^{-0.2}} = 793.41.$$

(7) loss elimination ratio:

This ratio calculates the difference between losses with and without a combination of deductible, limit, coinsurance and inflation.

$$LER = \frac{E(X) - E(Y^L)}{E(X)}.$$

Ex: loss X follow an exponential distribution with mean ~~\$~~ 1,000. we suppose that the policy is subject to a deductible of 500. Calculate the loss elimination ratio.

- $E(X) = \frac{1}{\lambda} = 1{,}000.$
- $Y^L = X - X \wedge d = (X-d)_+ .$

$$EY^L = EX - E(X\wedge d).$$
$$= \frac{1}{\lambda} - \frac{1}{\lambda}(1-e^{-\lambda d}).$$
$$= \frac{1}{\lambda} e^{-\lambda d}.$$

- $$LER = \frac{\frac{1}{\lambda} - \frac{1}{\lambda}e^{-\lambda d}}{1/\lambda} = 1 - e^{-\lambda d} = 1 - e^{-0.5}$$

$$= 0.393 = 39.3\%.$$

[For example if insurance company pays 2,000 without deductible, but with deductible 60.7% of 2,000 = 1214]

(8) Impact of deductible on the number of payments:

- Collective risk model:
  - N number of insurance policies in a period of times.
  - $X_1, X_2, \dots$ iid.
  - $S = X_1 + \dots + X_N$.
- we suppose that each X_i is subject to a deductible d.

N^L = number of cost-per loss.

N^P = number of cost per payment.

- $N^L = N$.
- we suppose: $X_1 = 100$, $X_2 = 150$, $X_3 = 120$, $X_4 = 130$.

$\underline{d = 125}$: $Y_1^L = 0$, $Y_2^L = 25$, $Y_3^L = 0$, $Y_4^L = 5$.

$\cancel{Y_1^P}$ $Y_2^P = 25$ $\cancel{Y_3^P}$, $Y_4^P = 5$.

- $N^P = \sum_{i=1}^{N} 1_{(X_i \geq d)} \doteq \sum_{i=1}^{N} Z_i$.
- $m_{N^P}(t) = m_N\left(\ln\left(m_Z(t)\right)\right)$, $Z = 1_{(X \geq d)}$

$Z \sim$ Bernoulli (p), $p = P(X \geq d) = S_X(d)$.

$$\boxed{m_{N^P}(t) = m_N\left(\ln\left(1 - p + pe^t\right)\right)}.$$

Ex: $N \sim Poi(\lambda) \longrightarrow N^P \sim Poi(\lambda p)$.

$$m_{N^P}(t) = e^{\lambda(1 - p + pe^t - 1)} = e^{\lambda p(e^t - 1)}.$$