Chapter ①: Distributions: Properties and characteristics.

① Median, Mode, Percentiles and Quartiles:

* Median: let X a random variable. The median of X is the number M such that:

$$P(X \leq M) \geq 0.5 \; ; \; P(X \geq M) \geq 0.5 .$$

For continuous random variables, we have

$$P(X \leq M) = P(X \geq M) = 0.5 .$$

Example ①: Consider:

x	0	1	2	3	4	5
$f(x)$	0.35	0.2	0.15	0.15	0.10	0.05
$F(x)$	0.35	0.55	0.7	0.85	0.95	1

Find the median of X?

$$P(X \leq 1) = F(1) = 0,55 \geq 0.5 .$$
$$P(X \geq 1) = 1 - F(0) = 0,65 \geq 0.5 .$$

$$\text{median}(X) = 1 .$$

Example ②: let X a Uniform (a,b). find the median of X.

$$F_X(M) = \frac{1}{2} \iff \frac{M-a}{b-a} = \frac{1}{2} .$$

$$\iff M = a + \frac{1}{2}(b-a) = \frac{a+b}{2} .$$

$$\text{median}(X) = a + \frac{1}{2}(b-a) = \frac{a+b}{2} .$$

* mode : the mode of X is the value that maximizes the mass function for discrete random variables, or the density function for continuous random variables.

Example ①: let $X \sim Geo(\frac{1}{2})$; $f(x) = \left(\frac{1}{2}\right)^x$, $x = 1, \ldots$

$$g(x) = \left(\frac{1}{2}\right)^x, \quad x \in \mathbb{R}.$$

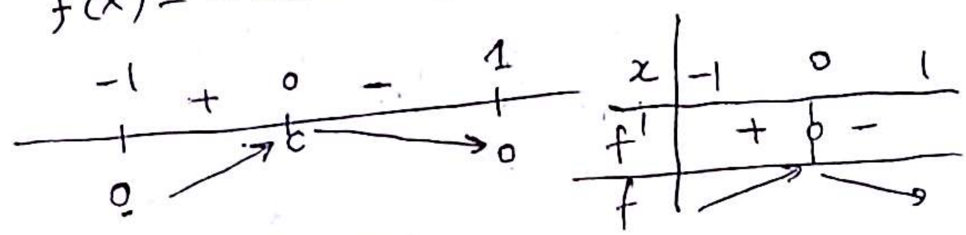
$$g'(x) = \left(\frac{1}{2}\right)^x \ln\left(\frac{1}{2}\right) < 0, \quad g \text{ is decreasing.}$$

$g(1)$ is the maximum.

$$\text{mode}(X) = 1.$$

Example ②: let X with $f_X(x) = c(1 - x^2)$; $-1 \le x \le 1$. $(c = 3/4)$.

$$f'(x) = -2cx = 0 \Leftrightarrow x = 0$$



$$\text{mode}(X) = 0.$$

* Percentiles:

let $0 < p < 1$. The $100p^{th}$ percentile (p^{th} quantile) is the number x such that:

$$P(X < x) \le p \le P(X \le x).$$

Example: let the mass function of X:

x	-1	0	1	2	4
$f(x)$	0.1	0.2	0.15	0.3	0.25
$F(x)$	0.1	0.3	0.45	0.75	1

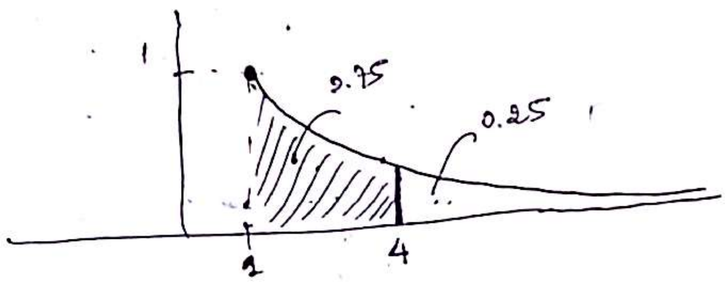
Compute the 25^{th} percentile (0.25 quantile)?

$$P(X < 0) = 0.1 \le 0.25.$$
$$P(X \le 0) = 0.3 \ge 0.25.$$

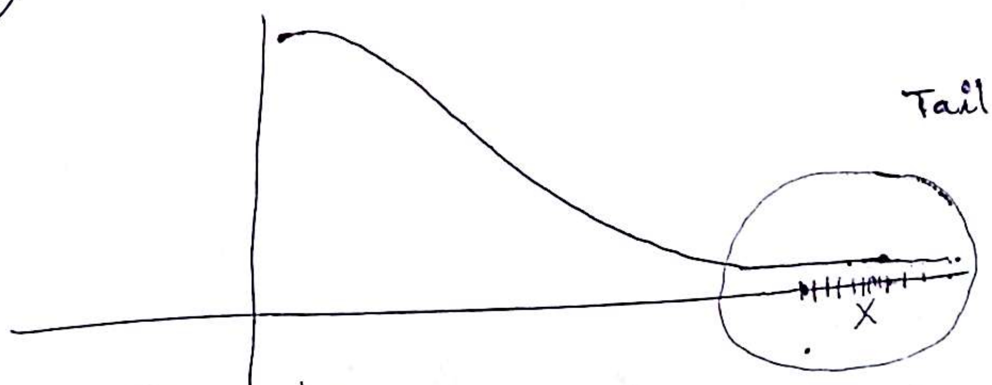
$$P_{25} = 0.$$

Example: let X with $f(x) = \frac{1}{x^2}$; $x \geq 1$.
Compute the 75^{th} percentile (0.75 quantile)?

$$P(X \le x) = 0.75 = \int_1^x \frac{1}{t^2}\, dt.$$
$$= -\frac{1}{t}\Big|_1^x = 1 - \frac{1}{x}.$$
$$\frac{1}{x} = 1 - 0.75 = 0.25$$
$$x = \frac{1}{0.25} = 4.$$



(2) Tail weight measures:



* Moments:
A distribution $f_X(x)$ is said to be
→ light-tailed if $E(X^k) < \infty$ for all $k > 1$,
→ heavy-tailed if $E(X^k)$ does not exists for ~~all~~ some k.

Example: (1) $X \sim Exp(\lambda)$.

$$E(X^k) = \frac{\Gamma(k+1)}{\lambda^k} < \infty .$$

X is light-tailed.

(2) $X \sim Pareto(\alpha, x_0)$.

$$f(x) = \frac{\alpha x_0^{\alpha}}{x^{\alpha+1}}, \quad x \geq x_0 .$$

$$E(X^k) = \int_{x_0}^{\infty} x^k \frac{\alpha x_0^{\alpha}}{x^{\alpha+1}} \, dx .$$

$$= C \, x^{k-\alpha} \Big|_{x_0}^{\infty} = \infty , \quad k > \alpha$$

$$C = \frac{\alpha x_0^{\alpha}}{k-\alpha}$$

Pareto-distribution is heavy-tailed.

* Survival function:

. the survival function $S(x) = P(X > x)$.

we say that a distribution X is:

→ lighter-tailed than Y if:

$$\lim_{x \to \infty} \frac{S_X(x)}{S_Y(x)} = \lim_{x \to \infty} \frac{f_X(x)}{f_Y(x)} = 0 .$$

→ heavier-tail than Y if:

$$\lim_{x \to \infty} \frac{S_X(x)}{S_Y(x)} = \infty .$$

Example: $X \sim \text{Inv-Gamma}(\alpha, \beta)$,
$Y \sim \text{Inv-Pareto}(\lambda, \tau)$.

$$f_X(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\beta/x}, \quad \alpha > 1.$$

$$f_Y(x) = \frac{\tau \lambda x^{\tau-1}}{(x+\lambda)^{\tau+1}} \quad \beta, \lambda, \tau > 0.$$

Compare the tail weight for X and Y?

$$\lim_{x\to\infty} \frac{f_X(x)}{f_Y(x)} = \lim_{x\to\infty} C \frac{x^{-\alpha-1} e^{-\beta/x} (x+\lambda)^{\tau+1}}{x^{\tau-1}}$$

$$= C \lim_{x\to\infty} x^{-\tau-\alpha} e^{-\beta/x} (x+\lambda)^{\tau+1}.$$

$$= C \lim_{x\to\infty} \frac{(x+\lambda)^{\tau+1}}{x^{\tau+\alpha}} = 0.$$

X is lighter tail than Y.

* Hazard rate fonction:

$$h(x) = -\frac{S'(x)}{S(x)}.$$

we say that a distribution is:

- → light-tailed if h is increasing.
- → heavy-tailed if h is decreasing.

Example: $X \sim \text{Pareto}(1,1)$, $f(x) = \frac{1}{x^2}$, $x \geq 1$.

$$S(x) = \int_x^\infty \frac{1}{t^2} dt = -\frac{1}{t}\Big|_x^\infty = \frac{1}{x}.$$

$$h(x) = -\frac{-1/x^2}{1/x} = \frac{1}{x} \text{ is decreasing.}$$

X is heavy-tailed.

Example: $X \sim \exp(\lambda)$, $S(x) = e^{-\lambda x}$.

$$h(x) = -\frac{-\lambda e^{-\lambda x}}{e^{-\lambda x}} = \lambda.$$

X is medium tailed.

* Mean excess loss function:

$$e(x) = E\big((X-x)\,|\,X>x\big) = \frac{E(X)-E(X\wedge x)}{1-F(x)}.$$

We say that a distribution is:

→ light-tailed if $e(x)$ is decreasing

→ heavy-tailed if $e(x)$ is increasing.

Example: $X \sim Exp(\lambda)$.

$$E(X\wedge x) = \int_0^\infty (t\wedge x)\,\lambda e^{-\lambda t}\,dt.$$

$$= \int_0^x t\,\lambda e^{-\lambda t}\,dt + \int_x^\infty x\,\lambda e^{-\lambda t}\,dt$$

$u = t$
$dv = \lambda e^{-\lambda t}dt$
$du = dt$
$v = -e^{-\lambda t}$

$$= -e^{-\lambda t}\Big|_0^x + \int_0^x e^{-\lambda t}\,dt + x \quad S(x)$$

$$= 1-e^{-\lambda x} + \frac{1}{\lambda}(1-e^{-\lambda x}) + x\,e^{-\lambda x}.$$

$$e(x) = \frac{\frac{1}{\lambda} - (1-e^{-\lambda x}) - \frac{1}{\lambda}(1-e^{-\lambda x}) - x e^{-\lambda x}}{e^{-\lambda x}}$$

$$= \left(\cancel{\tfrac{1}{\lambda}} - 1 - \cancel{\tfrac{1}{\lambda}}\right)e^{\lambda x} + \left(1+\frac{1}{\lambda}-x\right).$$

$$= -e^{\lambda x} + \left(1+\frac{1}{\lambda}-x\right) \text{ is decreasing.}$$

X is light-tailed.

③ Risk measures:

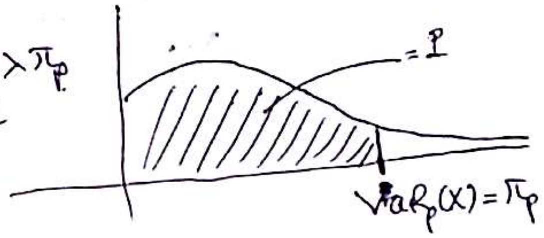
- Value-at-risk measure:

Suppose a risk X. We define the value-at-risk of X at the $100p\%$ level by:

$$VaR_p(X) = \pi_p(X) = 100p\% \text{ percentile}$$
$$= p \text{ quantile of } X.$$

Example: $X \sim Exp(\lambda)$. Find $VaR_p(X)$.

$$F_X(\pi_p) = p$$
$$1 - e^{-\lambda \pi_p} = p \Leftrightarrow 1-p = e^{-\lambda \pi_p}$$
$$-\lambda \pi_p = \ln(1-p)$$
$$\pi_p = -\frac{1}{\lambda}\ln(1-p).$$



Example: $X \sim Pareto(\alpha, \theta)$, $\cancel{f(x) = \alpha\left(\frac{\theta}{\theta+x}\right)^{\alpha-1}}$

Find π_p.

$$\cancel{F(x) = \int_1^x \alpha\theta\left(\frac{\theta}{\theta+t}\right)^{\alpha-1} dt = -\left(\frac{\theta}{\theta+t}\right)^{\alpha}\Big|_1^x}$$

$$f(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} \quad ; \quad x > 0.$$

$$F(x) = \int_0^x \frac{\alpha\theta^\alpha}{(t+\theta)^{\alpha+1}} dt = \frac{\alpha\theta^\alpha (t+\theta)^{-\alpha}}{-\alpha}\Big|_0^x$$

$$\boxed{\int (t+\theta)^\beta dt = \frac{(t+\theta)^{\beta+1}}{\beta+1}}$$

$$= -\theta^\alpha (x+\theta)^{-\alpha} + 1$$
$$= 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha.$$

$$F(x) = p$$
$$1 - \left(\frac{\theta}{x+\theta}\right)^\alpha = p \Leftrightarrow \left(\frac{\theta}{x+\theta}\right)^\alpha = 1-p$$
$$\frac{\theta}{x+\theta} = (1-p)^{1/\alpha} \Leftrightarrow \frac{x+\theta}{\theta} = (1-p)^{-1/\alpha}$$
$$\pi_p = \frac{\theta}{(1-p)^{1/\alpha}} - \theta.$$

- Tail-value-at-risk :

The tail-value-at-risk of X at $100p\%$ level is:

$$TVaR_p(X) = E(X \mid X \geq VaR_p(X)).$$

$$= \frac{\int_{\pi_p}^{\infty} x f_X(x)\,dx}{1-p}.$$

Example: $X \sim Exp(\lambda)$. Compute $TVaR_p(X)$?

$$\int_{\pi_p}^{\infty} \lambda x e^{-\lambda x}\,dx = -x e^{-\lambda x}\Big|_{\pi}^{\infty} + \int_{\pi}^{\infty} e^{-\lambda x}\,dx.$$

$$= -x e^{-\lambda x} - \frac{1}{\lambda} e^{-\lambda x}\Big|_{\pi}^{\infty} = \pi e^{-\lambda \pi} + \frac{1}{\lambda} e^{-\lambda \pi}$$

$$= -\frac{1}{\lambda}\ln(1-p)(1-p) + \frac{1}{\lambda}(1-p).$$

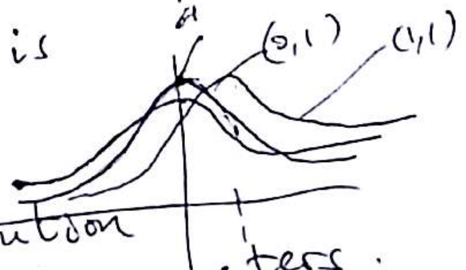
$$TVaR_p(X) = \frac{1}{\lambda}(1 - \ln(1-p)).$$

4 Parametric and Scale distributions:

- A parametric distribution is any distribution determined completely by a set of parameters.

Exs: Poisson(λ), Normal(μ, σ), Gamma(α, β), ...

- A scale distribution X is a distribution such that cX has the same distribution with possible different parameters.



$(c>0)$

Ex: Show that the Pareto distribution is a scale distribution.

$c>0$, $X \sim$ Pareto (α, θ) ; $Y = cX \sim$

$$F_Y(y) = P(Y \leq y) = P(cX \leq y)$$
$$= P(X \leq y/c) = F_X(y/c)$$
$$= 1 - \left(\frac{\theta}{y/c + \theta}\right)^{\alpha} = 1 - \left(\frac{c\theta}{y + c\theta}\right)^{\alpha}.$$

$Y = cX \sim$ Pareto $(\alpha, c\theta)$.

(5) Discrete mixture distributions:

A random variable X is a k-point mixture of $X_1, \dots, X_k$ if:

$$F_X(t) = a_1 F_{X_1}(t) + a_2 F_{X_2}(t) + \dots + a_k F_{X_k}(t).$$

$a_1, \dots, a_k \geq 0$, $\sum_{i=1}^{k} a_i = 1$.

$$E(X^m) = a_1 E(X_1^m) + \dots + a_k E(X_k^m)$$

$$m_X(t) = a_1 m_{X_1}(t) + \dots + a_k m_{X_k}(t).$$

(6) Data-dependent distributions:

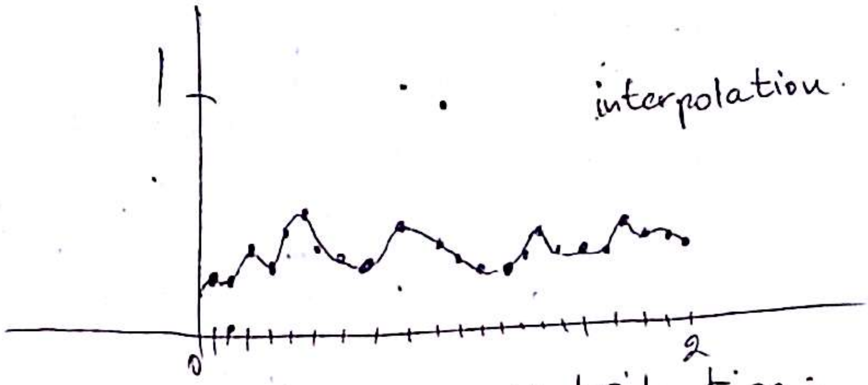
it is a distribution, constructed from data.

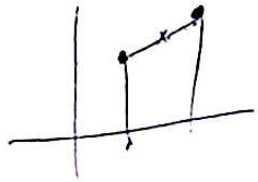
- empirical distribution is the distribution, which gives to any value in the data, its frequency.

Example: losses of an insurance company are: 49, 50, 50, 50, 60, 75, 80, 120, 230.

let X a random variable, taking these values. Find the mass function of X?

x	49	50	60	75	80	120	230
$f_X(x)$	$\frac{1}{9}$	$3/9$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$





- kernel Smoothed distribution:

Given an empirical distribution $p_n(x) = \frac{\#(x)}{n}$.

A kernel Smoothed distribution is given by:

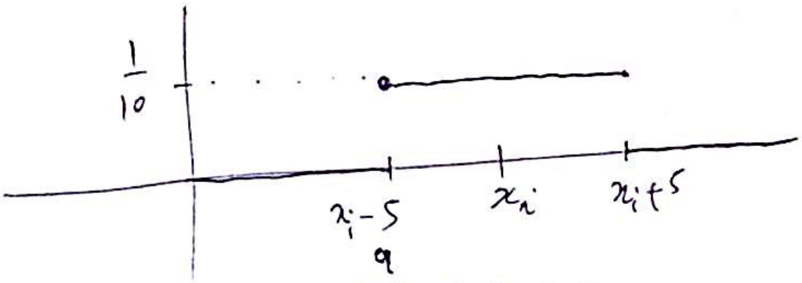
$$f(x) = \sum_{i=1}^{n} p_n(x)\, k_i(x),$$

$k_i(x)$ = kernel smoothed density function.

Example: Data: 49, 50, 50, 50, 60, 75, 80, 120, 230.

Develop a kernel smoothed distribution for this data, using a kernel density function

$$k_i(x) = \begin{cases} 1/10 & x_i - 5 \le x \le x_i + 5 \\ 0 & \text{o.w.} \end{cases}$$



$$f_X(x) = \sum_{i=1}^{9} \frac{1}{9} k_i(x).$$