# Power Series Solutions

Recall, a power series in powers of $(x - x_0)$ is an infinite sum on the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

for example:

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^{n+1}}, \sum_{n=1}^{\infty} \frac{(3x+1)^{n-1}}{\sqrt{n+1}}, \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

The set of all values of $x$ for which a power series converges is called the interval of convergence of the series.

Suppose that the power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

has a positive radius of convergence, that is there is a positive number $\rho$ such that the power series converges for all

$$x \in I = (x_0 - \rho, x_0 + \rho)$$

If $\sum_{n=1}^{\infty} a_n (x - x_0)^n = 0$ for all $x \in I$, then

$$a_n = 0 \quad for\ all\ n = 0, 1, \ldots$$

Two power series $\sum_{n=i}^{\infty} a_n (x - x_0)^n$ and $\sum_{m=j}^{\infty} b_n (x - x_0)^m$ can be combined by addition or subtraction provided that:

(i) they start with the same power of $x$

(ii) their summation indices start at the same value.

**Definition:**

A function $f$ is said to be analytic at a point $x_0$ if it can be represented by a power series on the form $\sum_{n=0}^{\infty} a_n (x - x_0)^n$, with a positive radius of convergence.

For example, $\sin x$ and $e^x$ are analytic functions everywhere, while $\frac{1}{x}$ is analytic except at $x = 0$.

**Remark.** Every polynomial is analytic everywhere, and every rational function is analytic except at the zeros of its denominator .

Consider the second order differential equation

$$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=0. \qquad (1)$$

Dividing both sides by $a_2(x)$, Eq.(1) can be written as

$$\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=0,$$

where $p(x)=\dfrac{a_1(x)}{a_2(x)},\ \ q(x)=\dfrac{a_0(x)}{a_2(x)}.$

A point $x_0$ is called an ordinary point of equation (1) if both $p(x)\ \ and\ \ q(x)$ are analytic at $x_0$.

A point which is not an ordinary point of the differential equation is called a singular point of the equation.

The point $x_0 = 0$ is an ordinary point of the DE

$$\frac{d^2 y}{dx^2} + (e^x)\frac{dy}{dx} + (\sin x)y = 0.$$

Because both functions $p(x) = e^x$ *and* $q(x) = \sin x$ are analytic at $x_0$

since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ , $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

and these series have the interval of convergence $(-\infty, \infty)$ ,

while $x_0 = 0$ is a singular point of the DE

$$\frac{d^2 y}{dx^2} + (\ln x)\frac{dy}{dx} + x^2 y = 0.$$

A singular point $x = x_0$ is called regular singular point of Eq.(1) if both $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytic at $x_0$. A singular point which is not regular is said to be irregular singular point.

## Example

Determine the ordinary points, the regular singular points and irregular singular points of the DE:

$$(x^4 - x^2)y'' + (2x+1)y' + x^2(x+1)y = 0$$

Let us put the equation on the form

$$y'' + p(x)y' + q(x)y = 0, \quad where$$

$$p(x) = \frac{2x+1}{x^4 - x^2} = \frac{2x+1}{x^2(x-1)(x+1)}, \; and$$

$$q(x) = \frac{x^2(x+1)}{x^4 - x^2} = \frac{x^2(x+1)}{x^2(x-1)(x+1)} = \frac{1}{x-1},$$

*after canceling common factors* .

Thus all real numbers except 0,1,-1 are ordinary points and 0,-1,1 are singular points.

Now,

$$(x - x_0)\,p(x) = (x - x_0)\frac{2x+1}{x^2(x-1)(x+1)}, \; and$$

$$(x - x_0)^2 q(x) = \frac{(x - x_0)^2}{x-1} \; .$$

At $x_0 = 0$ we have

$$(x - x_0)\,p(x) = \frac{2x+1}{x(x-1)(x+1)},\ and\ (x - x_0)^2 q(x) = \frac{x^2}{x-1}\ .$$

The first function is discontinuous at $x = 0$, therefore this is irregular singular point.

At $x_0 = 1$ we have

$$(x - x_0)\,p(x) = \frac{2x+1}{x^2(x+1)},\ and\ (x - x_0)^2 q(x) = x - 1\ .$$

Both functions are analytic at $x = 1$, thus it is regular singular point.

At $x_0 = -1$ we have

$$(x - x_0)\,p(x) = \frac{2x+1}{x^2(x-1)},\ and\ (x - x_0)^2 q(x) = \frac{(x+1)^2}{x-1}\ .$$

Both functions are analytic at $x = -1$, thus it is regular singular point.

# Theorem:

If $x_0$ is an ordinary point of the DE

$$\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=f(x),$$

then there are two linearly independent power series solutions of this equation on the form

$$y_1=\sum_{n=0}^{\infty}a_n(x-x_0)^n,\quad y_2=\sum_{n=0}^{\infty}b_n(x-x_0)^n,$$

with an interval of convergence centered at $x_0$ and has a positive radius of convergence.

# Example

Find the general solution in power series form for the differential equation

$$y' - 2xy = 0, \qquad (1)$$

about the ordinary point $x_0 = 0$.

Solution. Assume that the solution is given by

$$y = \sum_{n=0}^{\infty} c_n x^n \Rightarrow y' = \sum_{n=1}^{\infty} n c_n x^{n-1},$$

using the values of $y$ and $y'$ in equation (1) we get

$$\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} 2 c_n x^{n+1} = 0.$$

To make the powers of $\chi$ similar in both series, put

$n-1=k$ in the firs series and $n+1=k$ in the second one , to get

$$\sum_{k=0}^{\infty}(k+1)\,c_{k+1}\,x^k-\sum_{k=1}^{\infty}2c_{k-1}\,x^k=0,$$

$$\Rightarrow\ c_1+\sum_{k=1}^{\infty}(k+1)\,c_{k+1}\,x^k-\sum_{k=1}^{\infty}2c_{k-1}\,x^k=0,$$

$$\Rightarrow\ c_1+\sum_{k=1}^{\infty}[(k+1)\,c_{k+1}-2c_{k-1}]x^k=0.$$

Since this true for all values of $\chi$ we get

$$c_1=0,\quad and$$

$$c_{k+1}=\tfrac{2}{k+1}c_{k-1},\ \ for\ \ all\ \ k=1,2,3,\ldots \qquad (2)$$

From the recurrence relation (2) we obtain

$$k=1 \Rightarrow c_2 = c_0,$$

$$k=2 \Rightarrow c_3 = \tfrac{2}{3}c_1 = 0,$$

$$k=3 \Rightarrow c_4 = \tfrac{1}{2}c_2 = \tfrac{1}{2}c_0,$$

$$k=4 \Rightarrow c_5 = \tfrac{2}{5}c_3 = 0,$$

$$k=5 \Rightarrow c_6 = \tfrac{1}{3}c_4 = \tfrac{1}{6}c_0,$$

.....

Now, from the assumption we have

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + \ldots$$

Or

$$y = c_0 + c_0 x^2 + \tfrac{1}{2} c_0 x^4 + \tfrac{1}{6} c_0 x^6 + \ldots$$

$$= c_0 [1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3} + \ldots]$$

$$= c_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} = e^{2x}.$$

## Example

Find the general solution of the differential equation

$$(x-1)y'' + y' = 0, \qquad (1)$$

about the ordinary point $x_0 = 0$.

Solution. First let us write equation (1) as

$$xy''-y''+y'=0, \qquad (2)$$

Assume $y=\sum_{n=0}^{\infty} c_n x^n \Rightarrow y'=\sum_{n=1}^{\infty} n c_n x^{n-1}, \; y''=\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2},$

using the values of $y$, $y'$ and $y''$ in equation (2) we get

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^{n-1} = 0.$$

To make the powers of $x$ similar in all series, put

$n-1=k$ in the firs series, $n-2=k$ in the second one , and

$n-1=k$ in the last one, to get

$$\sum_{k=1}^{\infty} k(k+1)\, c_{k+1}\, x^k - \sum_{k=0}^{\infty} (k+1)(k+2) c_{k+2}\, x^k + \sum_{k=0}^{\infty} (k+1) c_{k+1}\, x^k = 0,$$

$$\Rightarrow \sum_{k=1}^{\infty} k(k+1)\, c_{k+1}\, x^k - 2c_2 - \sum_{k=1}^{\infty} (k+1)(k+2) c_{k+2}\, x^k + c_1 + \sum_{k=0}^{\infty} (k+1) c_{k+1}\, x^k = 0,$$

$$\Rightarrow c_1 - 2c_2 + \sum_{k=1}^{\infty} [(k+1)(k+1)\, c_{k+1} - (k+1)(k+2) c_{k+2}\, ] x^k = 0,$$

$$c_1 - 2c_2 = 0 \Rightarrow c_2 = \tfrac{1}{2} c_1, \quad and$$

$$c_{k+2} = \tfrac{k+1}{k+2} c_{k+1}, \quad for \quad all \quad k = 1,2,3,\ldots \qquad (3)$$

From the recurrence relation (3) we obtain

$$k = 1 \Rightarrow c_3 = \tfrac{2}{3} c_2 = \tfrac{1}{3} c_1,$$

$$k = 2 \Rightarrow c_4 = \tfrac{3}{4} c_3 = \tfrac{1}{4} c_1,$$

$$k = 3 \Rightarrow c_5 = \tfrac{1}{5} c_1,$$

$$\ldots$$

Now, from our assumption we have

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \ldots$$

$$= c_0 + c_1 x + \tfrac{1}{2} c_1 x^2 + \tfrac{1}{3} c_1 x^3 + \tfrac{1}{4} c_1 x^4 + \tfrac{1}{5} c_1 x^5 + \ldots$$

$$= c_0 + c_1 [x + \tfrac{1}{2} x^2 + \tfrac{1}{3} x^3 + \tfrac{1}{4} x^4 + \tfrac{1}{5} x^5 + \ldots]$$

$$= c_0 + c_1 \sum_{n=1}^{\infty} \tfrac{x^k}{k} \, .$$

Example

Find the general solution in power series form for the differential equation

$$y''-xy=2+3x, \qquad (1)$$

about the ordinary point $x_0=0.$

Solution. Assume that the solution is given by

$$y=\sum_{n=0}^{\infty}c_n x^n \Rightarrow y'=\sum_{n=1}^{\infty}n c_n x^{n-1}, \quad y''=\sum_{n=2}^{\infty}n(n-1)c_n x^{n-2},$$

using the values of $y$ and $y''$ in equation (1) we get

$$\sum_{n=2}^{\infty}n(n-1)c_n x^{n-2}-\sum_{n=0}^{\infty}c_n x^{n+1}=2+3x.$$

Put $n-2=k$ in the firs series and $n+1=k$ in the second one , to get

$$\sum_{n=0}^{\infty}(k+1)(k+2)c_{k+2}\,x^k - \sum_{k=1}^{\infty}c_{k-1}\,x^k = 2+3x$$

Or

$$c_2 + (6c_3 - c_0)x + \sum_{k=2}^{\infty}[(k+1)(k+2)c_{k+2} - c_{k-1}]x^k = 2+3x$$

Which implies that

$$c_2 = 2,$$

$$6c_3 - c_0 = 3 \Rightarrow c_3 = \tfrac{1}{6}c_0\,, \quad and$$

$$(k+1)(k+2)c_{k+2} - c_{k-1} = 0, \quad for \quad k \geq 2. \qquad (2)$$

From the recurrence relation (2) we obtain

$$c_{k+2} = \frac{1}{(k+1)(k+2)} c_{k-1}, \; k = 2,3,4,...$$

Which implies

$$k = 2 \Rightarrow c_4 = \tfrac{1}{12} c_1,$$

$$k = 3 \Rightarrow c_5 = \tfrac{1}{20} c_2 = \tfrac{1}{10},$$

$$k = 4 \Rightarrow c_6 = \tfrac{1}{30} c_3 = \tfrac{1}{180} c_0 \;,$$

and so on. Now, from the assumption we have

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + ...$$

$$= c_0 + c_1 x + 2x^2 + \tfrac{1}{6} c_0 x^3 + \tfrac{1}{12} c_1 x^4 + \tfrac{1}{10} x^5 + \tfrac{1}{180} c_0 x^6 + ...$$

$$= c_0 (1 + \tfrac{1}{6} x^3 + \tfrac{1}{180} x^6 + ...) + c_1 (x + \tfrac{1}{12} x^4 + ...) + 2x^2 + \tfrac{1}{10} x^5 + ...$$

Example

Find the general solution in power series form for the differential equation

$$y''-xy=0, \qquad (1)$$

about the ordinary point $x_0=2$.

Solution. Assume that the solution is given by

$$y=\sum_{n=0}^{\infty}c_n(x-2)^n \;\Rightarrow\; y'=\sum_{n=1}^{\infty}n\,c_n(x-2)^{n-1},\;\; y''=\sum_{n=2}^{\infty}n(n-1)\,c_n(x-2)^{n-2}.$$

Now, let us write equation (1) as

$$y''-(x-2)y-2y=0, \qquad (2)$$

using the values of $y$ and $y''$ in (2) we get

$$\sum_{n=2}^{\infty} n(n-1)c_n (x-2)^{n-2} - \sum_{n=0}^{\infty} c_n (x-2)^{n+1} - \sum_{n=0}^{\infty} 2c_n (x-2)^n = 0.$$

Putting $n-2=k$ in the firs series, $n+1=k$ in the second one and $n=k$ in the last one we get

$$\sum_{n=0}^{\infty} (k+1)(k+2)c_{k+2} (x-2)^k - \sum_{k=1}^{\infty} c_{k-1} (x-2)^k - \sum_{k=0}^{\infty} 2c_k (x-2)^k = 0,$$

or

$$2c_2 + \sum_{n=1}^{\infty} (k+1)(k+2)c_{k+2} (x-2)^k - \sum_{k=1}^{\infty} c_{k-1} (x-2)^k - 2c_0 - \sum_{k=1}^{\infty} 2c_k (x-2)^k = 0,$$

$$2(c_2 - c_0) + \sum_{n=1}^{\infty} [(k+1)(k+2)c_{k+2} - c_{k-1} - 2c_k](x-2)^k = 0$$

It follows that

$$c_2 = c_0, \quad and$$

$$c_{k+2} = \frac{2c_k - c_{k-1}}{(k+1)(k+2)}, \quad for \quad k = 1,2,3,... \qquad (3)$$

which implies that

$$k = 1 \Rightarrow c_3 = \frac{2c_1 - c_0}{6},$$

$$k = 2 \Rightarrow c_4 = \frac{2c_2 - c_1}{12} = \frac{2c_0 - c_1}{12},$$

….

Using the values of these coefficients in the assumption we have

$$y = \sum_{n=0}^{\infty} c_n (x-2)^n = c_0 + c_1(x-2) + c_2(x-2)^2 + c_3(x-2)^3 + ...$$

$$y = c_0 + c_1(x-2) + c_0(x-2)^2 + \tfrac{1}{6}(2c_1 - c_0)(x-2)^3 + \ldots$$

$$= c_0[1 + (x-2)^2 - \tfrac{1}{6}c_0(x-2)^3 + \ldots.] + c_1[(x-2) + \tfrac{1}{3}(x-2) + \ldots].$$

## Remark

We can use the following change of variables to transform the ordinary point $x_0 = 2$ to the origin $t_0 = 0$ and proceed as before:

$$t = x - 2 \Rightarrow \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt},$$

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}\frac{dt}{dx} = \frac{d^2y}{dt^2}.$$

Thus the differential equation becomes

$$\frac{d^2y}{dt^2} - (t+2)y = 0.$$

Example

Find the solution of the initial value problem

$$y''+x^2 y'-y=0, \quad y(0)=1, \quad y'(0)=-2, \qquad (1)$$

using the power series method about the ordinary point $x_0=0$.

Solution. Assume that the solution is given by

$$y=\sum_{n=0}^{\infty} c_n x^n \Rightarrow y'=\sum_{n=1}^{\infty} n c_n x^{n-1}, \quad y''=\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}.$$

using the values of $y$, $y'$ and $y''$ in (1) we obtain:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^{n+1} - \sum_{n=0}^{\infty} c_n x^n = 0.$$

Put $n-2=k$ in the firs series, $n+1=k$ in the second, and $n=k$ in the last one , to get

$$\sum_{k=0}^{\infty}(k+1)(k+2)c_{k+2}\,x^k+\sum_{k=2}^{\infty}(k-1)c_{k-1}\,x^k-\sum_{k=0}^{\infty}c_k\,x^k=0,$$

$$\Rightarrow 2c_2+6c_3x+\sum_{k=2}^{\infty}(k+1)(k+2)c_{k+2}\,x^k+\sum_{k=2}^{\infty}(k-1)c_{k-1}\,x^k-c_0-c_1x-\sum_{k=2}^{\infty}c_k\,x^k=0,$$

$$\Rightarrow (2c_2-c_0)+(6c_3-c_1)x+\sum_{k=1}^{\infty}[(k+1)(k+2)c_{k+2}+(k-1)c_{k-1}-c_k]x^k=0,$$

$$\Rightarrow 2c_2-c_0=0\Rightarrow c_2=\tfrac{1}{2}c_0,$$

$$6c_3-c_1\Rightarrow c_3=\tfrac{1}{6}c_1,\quad and$$

$$(k+1)(k+2)c_{k+2}+(k-1)c_{k-1}-c_k=0,\quad for\ k\geq 2,$$

$$\Rightarrow c_{k+2}=\tfrac{c_k-(k-1)c_{k-1}}{(k+1)(k+2)},\ \ k=2,3,\ldots..$$

Hence, $k=2\Rightarrow c_4=\tfrac{1}{12}[c_2-c_1]=\tfrac{1}{12}[\tfrac{1}{2}c_0-c_1],$

$$K=3\Rightarrow c_5=\tfrac{1}{20}[c_3-2c_2]=\tfrac{1}{12}[\tfrac{1}{6}c_1-c_0],$$

Now, from the assumption we have

$$y=\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + \ldots$$

$$= c_0 + c_1 x + \tfrac{1}{2} c_0 x^2 + \tfrac{1}{6} c_1 x^3 + \tfrac{1}{12}(\tfrac{1}{2} c_0 - c_1) x^4 + \ldots$$

$$= c_0(1 + \tfrac{1}{2} x^2 + \tfrac{1}{24} x^4 + \ldots) + c_1(x + \tfrac{1}{6} x^3 - \tfrac{1}{12} x^4 + \ldots).$$

$$Since \quad y(0) = 1 \Rightarrow c_0 = 1,$$

$$and \quad y'(0) = -2 \Rightarrow c_2 = -2,$$

*hence*

$$y = (1 + \tfrac{1}{2} x^2 + \tfrac{1}{24} x^4 + \ldots) - 2(x + \tfrac{1}{6} x^3 - \tfrac{1}{12} x^4 + \ldots).$$