Systems of Linear D.Es.

Let $y = f(x)$ be a differentiable function. Then the derivatives $y', y'', y''', \ldots, y^{(n)}$ are also denoted by

$D, D^2, D^3, \ldots, D^n$, respectively. Hence the D.E.

$5y'' - 3y' + 2y = 0$ can be written as $(5D^2 - 3D + 2)y = 0.$

The expression $(5D^2 - 3D + 2)$ is called a linear differential operator, it operates on the function which follows. For example

$$
\begin{aligned}
&(D^2 - 3D + 2)(x^2 - 5x + 7) \\
&\quad = D^2(x^2 - 5x + 7) - 3D(x^2 - 5x + 7) + 2(x^2 - 5x + 7) \\
&\quad = 2 - 3(2x - 5) + 2(x^2 - 5x + 7) \\
&\quad = 2x^2 - 16x + 31.
\end{aligned}
$$

Suppose x *and* y are differentiable functions with respect to an independent variable t.

Then, any set of two or more equations involving derivatives of these functions is called a system of D.Es.

Let L_1, L_2, L_3 *and* L_4 be differential operators with constant coefficients, by solving the system

$$L_1(D)x + L_2(D)y = h_1(t),$$
$$L_3(D)x + L_4(D)y = h_2(t),$$

we mean to find two differentiable functions

$y = f(t)$ *and* $x = g(t)$ defined on some interval I and satisfy each equation in the system. One method of solution is to eliminate one of the dependent variables to get a single D.E. in the other one and solve it for this variable, next substitute the result in one of the original equations and solve the resulting equation to obtain the other dependent variable.

Example. Solve the following systems of D.Es.

(1) $\begin{cases} \frac{dx}{dt} - \frac{1}{2}\frac{dy}{dt} = -x + 2t - 1, \\ -2\frac{dx}{dt} + \frac{dy}{dt} = 6e^t. \end{cases}$

Solution. Multiplying the first equation by 2 then adding to the second one we get

$$x = 2t - 1 + 3e^t.$$

Using this result in the second equation we obtain

$$\frac{dy}{dt} = 6e^t + 2\frac{d}{dt}(2t - 1 + 3e^t) = 12e^t + 4 \Rightarrow y = 12e^t + 4t + c.$$

(2). $\begin{cases} \frac{dx}{dt} + \frac{dy}{dt} + 2y = 0, \\ \frac{dx}{dt} - 3x - 2y = 0. \end{cases}$

Solution. First write the system in operator notation:

$$\begin{cases} Dx + (D+2)y = 0, & (1) \\ (D-3)x - 2y = 0. & (2) \end{cases}$$

Multiplying the first equation by 2 and operating on the second one by $(D+2)$, then adding the results we get $(D^2 + D - 6)x = 0$ *or* $x'' + x' - 6x = 0$, which is a Hom. L.D.E. Hence the auxiliary equation is

$$m^2 + m - 6 = 0 \Rightarrow m = -3, 2,$$

$$\Rightarrow x = c_1 e^{-3t} + c_2 e^{2t}.$$

Using the value of x in the second equation we get

$$y = \tfrac{-3}{2}x + \tfrac{1}{2}\tfrac{dx}{dt} = \tfrac{-3}{2}(c_1 e^{-3t} + c_2 e^{2t}) + \tfrac{1}{2}(-3c_1 e^{-3t} + 2c_2 e^{2t})$$

$$= -3c_1 e^{-3t} - \tfrac{1}{2}c_2 e^{2t}.$$

(3). $\begin{cases} (D-4)x + D^2 y = 0, \\ (D+1)x + Dy = 0. \end{cases}$

Solution. Eliminating y by operating on the second equation by $-D$, then adding the result to the first equation we get $(D^2+4)x = 0$ *or* $x''+4x = 0,$ which is a Hom. L.D.E. Hence $x = c_1 \cos 2x + c_2 \sin 2x.$

Using this value in the second equation we obtain

$$Dy = -(c_1 \cos 2x + c_2 \sin 2x) - (-2c_1 \sin 2x + 2c_2 \cos 2x)$$

$$= -(c_1 + 2c_2)\cos 2x + (2c_1 - c_2)\sin 2x$$

$$\Rightarrow y = -\tfrac{1}{2}(c_1 + 2c_2)\cos 2x - \tfrac{1}{2}(2c_1 - c_2)\sin 2x + c_3.$$

(4). $\begin{cases} x' = y - 1, \\ y' = 3x + 2y, \\ y(0) = x(0) = 0. \end{cases}$

Solution. In operator notation the system becomes

$$\begin{cases} Dx - y = -1, \\ -3x + (D-2)y = 0. \end{cases}$$

Eliminating x we get $(D^2 - 2D - 3)y = -3.$

The aux. equation is $m^2 - 2m - 3 = 0 \Rightarrow m = -1, 3,$

hence $y_c = c_1 e^{-x} + c_2 e^{3x}.$

For y_p we have $g(t) = -3 \Rightarrow y_p = A = 1,$ therefore

$$y = y_c + y_p = c_1 e^{-x} + c_2 e^{3x} + 1.$$

Then from the second equation we have

$$x = \tfrac{1}{3}[-3c_1 e^{-x} + c_2 e^{3x} - 2].$$

Using the initial conditions we get

$$c_1 + c_2 = -1 \textit{ and } -3c_1 + c_2 = 2 \Rightarrow c_1 = \tfrac{-3}{4}, c_1 = \tfrac{-1}{4},$$

hence we get $y = \tfrac{-3}{4}e^{-x} - \tfrac{1}{4}e^{3x} + 1,$

$$x = \tfrac{1}{3}[\tfrac{9}{4}e^{-x} - \tfrac{1}{4}e^{3x} - 2].$$

(5) Solve the following system for y.

$$\begin{cases} (D^2 + D - 1)x + (D^2 - 3D + 2)y = 0, \\ (D+2)x + 2(D-2)y = 0. \end{cases}$$

Solution. Operating on the first equation by $(D+2)$ and on the second one by $(D^2 + D - 1)$ then subtract we get $(D^3 - D^2 - 2D)y = 0$. The aux. equation is

$$m^3 - m^2 - 2m = 0 \Rightarrow m = 0, -1, 2 \Rightarrow y = c_1 + c_2 e^{-t} + c_3 e^{2t}.$$

(6). $\begin{cases}(D+1)x-(D-1)y=2t, & (1)\\ -3x+(D+2)y=e^{-t}. & (2)\end{cases}$

Solution. To eliminate x multiply the first equation by 3 and operate on the second one by $(D+1)$ then add to get $(D^2+5)y=6t+(D+1)e^{-t}$,

$$or \quad y''+5y=6t. \qquad (3)$$

hence $y_c=c_1\cos\sqrt{5}t+c_2\sin\sqrt{5}t.$

Since $g(t)=6t \Rightarrow y_p=At+B$, using this in Equation (3) and comparing coefficients we get $A=\frac{6}{5}, =0 \Rightarrow y_p=\frac{6}{5}t$

therefore $y=y_c+y_p=c_1\cos\sqrt{5}t+c_2\sin\sqrt{5}t+\frac{6}{5}t,$

and from Eq.(2) we obtain

$$x=\tfrac{1}{3}[(2c_1+\sqrt{5}c_2)\cos\sqrt{5}t+(2c_2-\sqrt{5}c_1)\sin\sqrt{5}t-e^{-t}+\tfrac{12}{5}t+\tfrac{6}{5}].$$

Homework

(1) Solve the system

$$\begin{cases} x''' - 2y'' = 2\cos t, \\ x'' + 2y' = -2x - 2\cos t. \end{cases}$$

(2) Solve the following system for z.

$$\begin{cases} x' = 3x - y - z, \\ y' = x + y - z, \\ z' = x + y + z. \end{cases}$$