Variation of Parameters

Consider a second order L.D.E. on the standard form

$$y''+P(x)y'+Q(x)y = f(x), \qquad (1)$$

where P, Q and f are continuous on some interval I. Let $y_c = c_1y_1 + c_2y_2$ be the general solution of the associated homogeneous D.E. $y''+P(x)y'+Q(x)y = 0.$

Can we find two functions u_1, u_2 such that the $y_p = u_1y_1 + u_2y_2$ is a particular solution of Eq.(1)?

Let $y_p = u_1y_1 + u_2y_2$ be a solution of Eq.(1).

Then $y_p' = (u_1'y_1 + u_2'y_2) + (u_1y_1' + u_2y_2')$, for simplicity

Assume that $u_1' y_1 + u_2' y_2 = 0$ which implies that

$$y_p'' = \left(u_1' y_1' + u_2' y_2'\right) + \left(u_1 y_1'' + u_2 y_2''\right).$$

Substituting the values of y_p, y_p' *and* y_p'' in Eq.(1) we obtain $u_1' y_1' + u_n' y_2' = f(x).$

Using Cramer's rule to solve the two equations

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_n' y_2' &= f \end{aligned}$$

for u_1', u_2' we get

$$u'_1 = \frac{\begin{vmatrix} 0 & y_2 \\ f & y'_2 \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}} = \frac{W_1}{W}, \quad u'_2 = \frac{\begin{vmatrix} y_1 & 0 \\ y'_1 & f \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}} = \frac{W_2}{W},$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f & y'_2 \end{vmatrix}, \quad and \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y'_1 & f \end{vmatrix}.$$

Hence

$$u_1 = \int \frac{W_1}{W} dx, \quad and \quad u_2 = \int \frac{W_2}{W} dx.$$

Similarly, if we have the n^{th} order D.E.

$$a_n(x) y^{(n)} + ... + a_1(x) y' + a_0(x) y = g(x),$$

where $a_n, ..., a_1, a_0, g$ are continuous on some interval I and $a_n(x) \neq 0$ for all x in I.

Let $y_c = c_1 y_1 + ... + c_n y_n$ be the solution of the associated H.L.D.E.

$$a_n(x) y^{(n)} + ... + a_1(x) y' + a_0(x) y = 0,$$

then $y_p = u_1 y_1 + \ldots + u_n y_n$ where

$$u_i = \int \frac{W_i}{W} dx, \quad i = 1, \ldots, n,$$

$$W = \begin{vmatrix} y_1 & y_2 & & y_n \\ y'_1 & y'_2 & & y'_n \\ & \ldots & \ldots & \\ y_1^{(n-1)} & y_2^{(n-1)} & & y_n^{(n-1)} \end{vmatrix},$$ and W_i is obtained from W by

Replacing the i^{th} column in W by the column $\begin{pmatrix} 0 \\ \ldots \\ 0 \\ f \end{pmatrix}$,

where $f(x) = \frac{g(x)}{a_n(x)}$.

Example.1 Use variation of parameters to solve the D.E. $y''-y'-2y=4.$

Solution. The auxiliary equation is

$$m^2-m-2=0 \Rightarrow m=2,-1,$$

hence $y_c=c_1e^{2x}+c_2e^{x}=c_1y_1+c_2y_2.$

By variation of parameters we have $y_p=u_1y_1+u_2y_2.$

Also,

$$W=\begin{vmatrix} e^{2x} & e^{x} \\ 2e^{2x} & e^{x} \end{vmatrix}=-e^{3x},\quad W_1=\begin{vmatrix} 0 & e^{x} \\ 4 & e^{x} \end{vmatrix}=-4e^{x},\quad and\quad W_2=\begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & 4 \end{vmatrix}=4e^{2x},$$

hence

$$u_1=\int \tfrac{W_1}{W}dx=\int 4e^{-2x}dx=-2e^{-2x},\qquad and$$

$$u_2=\int \tfrac{W_2}{W}dx=\int -4e^{-x}dx=4e^{-x} \Rightarrow y_p=(-2e^{-2x})e^{2x}+(4e^{-x})e^{x}=2,$$

therefore the general solution is

$$y = y_c + y_p = c_1 e^{2x} + c_2 e^{x} + 2.$$

It is evident that solving this problem by using undetermined coefficients method will avoid a lot of computations.

Indeed, since $g(x) = 2 \Rightarrow y_p = A \Rightarrow y_p' = y_p'' = 0,$ using these values in the original D.E. and comparing coefficients we get $A = 2 \Rightarrow y_p = 2.$

Example 2. Solve the following differential equations

(1) $y'' + y = \sec x.$

Solution. The auxiliary equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm i,$$

hence $y_c = c_1 \cos x + c_2 \sin x = c_1 y_1 + c_2 y_2.$

By variation of parameters we have $y_p = u_1 y_1 + u_2 y_2.$

And

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1,\ W_1 = \begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix} = -\tan x,\ \textit{and}\ \ W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix} = 1,$$

Hence

$$u_1 = \int \tfrac{W_1}{W} dx = \int -\tan x dx = \ln|\cos x|, \quad \textit{and}$$

$$u_2 = \int \tfrac{W_2}{W} dx = \int dx = x \Rightarrow y_p = \cos x \ln|\cos x| + x,$$

And the general solution is

$$y = y_c + y_p = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x.$$

(2) $y''+y=\sec x\tan x.$

Solution. $y_c = c_1\cos x + c_2\sin x = c_1y_1 + c_2y_2.$

Now, by variation of parameters we have

$y_p = u_1y_1 + u_2y_2.$ But

$$W=1,\ W_1=\begin{vmatrix} 0 & \sin x \\ \sec x\tan x & \cos x \end{vmatrix} = -\tan^2 x,\ \textit{and}\ \ W_2=\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x\tan x \end{vmatrix} = \tan x,$$

Hence $u_1 = \int \frac{W_1}{W}dx = \int -\tan^2 xdx = x-\tan x,$

$$u_2 = \int \frac{W_2}{W}dx = \int \tan xdx = \ln|\sec x|,$$

$$\Rightarrow\ y_p = \cos x(x-\tan x)+\sin x\ln|\sec x|,$$

And the general solution is

$$y = y_c + y_p = c_1\cos x + C_2\sin x + x\cos x + \sin x\ln|\sec x|,\ \ C_2 = c_2 - 1.$$

(3) $y''+3y'+2y=\frac{1}{1+e^x}$.

Solution. The aux. eq. is $m^2+3m+2=0 \Rightarrow m=-2,-1,$

Hence $y_c=c_1e^{-2x}+c_2e^{-x}=c_1y_1+c_2y_2,$

therefore $y_p=u_1y_1+u_2y_2.$ But

$$W=\begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix}=e^{-3x},\quad W_1=\begin{vmatrix} 0 & e^{-x} \\ \frac{1}{1+e^x} & -e^{-x} \end{vmatrix}=\frac{-e^{-x}}{1+e^x},\quad W_2=\begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & \frac{1}{1+e^x} \end{vmatrix}=\frac{e^{-2x}}{1+e^x},$$

Hence $u_1=\int\frac{W_1}{W}dx=\int\frac{-e^{2x}}{1+e^x}dx=\ln(1+e^x)-1-e^x,$

$$u_2=\int\frac{W_2}{W}dx=\int\frac{e^x}{1+e^x}dx=\ln(1+e^x),$$

$$\Rightarrow y_p=e^{-2x}(\ln(1+e^x)-1-e^x)+e^{-x}\ln(1+e^x),$$

And the general solution is $y=y_c+y_p$.

Notice that some terms in y_p can be absorbed in y_c.

(4) $y''' + y' = \tan x.$

Solution. $y_c = c_1 + c_2 \cos x + c_3 \sin x = c_1 y_1 + c_2 y_2 + c_3 y_3.$

Then $y_p = u_1 y_1 + u_2 y_2 + u_3 y_3.$ But

$$W = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = -1, \quad W_1 = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \tan x & -\cos x & -\sin x \end{vmatrix} = \tan x,$$

$$W_2 = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \tan x & -\sin x \end{vmatrix} = -\sin x, \quad W_3 = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \tan x \end{vmatrix} = -\sin x \tan x,$$

Hence

$$u_1 = \int \tfrac{W_1}{W} dx = \int -\tan x dx = \ln|\cos x|,$$

$$u_2 = \int \tfrac{W_2}{W} dx = \int \sin x dx = -\cos x,$$

$$u_3 = \int \tfrac{W_3}{W} dx = \int \sin x \tan x dx = \int \tfrac{\sin^2 x}{\cos x} dx =$$

$$= \int \tfrac{1-\cos^2 x}{\cos x} dx = \int (\sec x - \cos x) dx = \ln|\sec x + \tan x| - \sin x,$$

$$\Rightarrow y_p = \ln|\cos x| + \cos x(-\cos x) + \sin x(\ln|\sec x + \tan x| - \sin x).$$

(3) $x^2y''-xy'+y=4x\ln x,\ x>0.$

Solution. This is a Cauchy-Euler equation, and

$$y_c=c_1x+c_2x\ln x=c_1y_1+c_2y_2,$$

therefore $y_p=u_1y_1+u_2y_2.$ But

$$W=\begin{vmatrix}x & x\ln x\\ 1 & 1+\ln x\end{vmatrix}=x,\quad W_1=\begin{vmatrix}0 & x\ln x\\ \frac{4\ln x}{x} & 1+\ln x\end{vmatrix}=-4\ln^2 x,\quad W_2=\begin{vmatrix}x & 0\\ 1 & \frac{4\ln x}{x}\end{vmatrix}=4\ln x,$$

hence

$$u_1=\int\frac{W_1}{W}dx=\int\frac{-4\ln^2 x}{x}dx=\frac{-4}{3}\ln^3 x,$$

$$u_2=\int\frac{W_2}{W}dx=\int\frac{4\ln x}{x}dx=2\ln^2 x,$$

$$\Rightarrow y_p=x\left(\frac{-4}{3}\ln^3 x\right)+x\ln x(2\ln^2 x)=\frac{2}{3}x\ln^3 x,$$

and the general solution is

$$y_c=c_1x+c_2x\ln x+\frac{2}{3}x\ln^3 x.$$

Homework

Use variation of parameters to solve the initial value problem:

$$4y'' - y = xe^{\frac{x}{2}}, \quad y(0) = 1, \; y'(0) = 0.$$