Nonhomogeneous Linear D.Es

Recall that a general n^{th} order L.D.E. is on the form

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\dots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x), \qquad (1)$$

where $a_0, a_1, \dots, a_n, g$ are continuous functions on some interval I *and* $a_n(x)\neq 0 \quad \text{for all} \quad x \text{ in } I.$

The general solution of Eq.(1) is on the form

$$y = y_c + y_p,$$

where y_c is the general solution of the associated Hom. D.E.

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\dots+a_1(x)\frac{dy}{dx}+a_0(x)y=0,$$

and y_p is a particular solution of Nonhom. E. Eq.(1).

Undetermined coefficients method

Consider an n^{th} order L.D.E. with constant coefficients

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_0 y = g(x), \qquad (1)$$

where $a_0, a_1, \ldots, a_n$ are constants.

We learned in Section 4.2 how can we determined y_c which is the general solution of the Hom. L.D. E. associated with Eq.(1) using the auxiliary equation:

$$a_n m^n + a_{n-1} m^{n-1} + \ldots + a_1 m + a_0 = 0.$$

Now, if $g(x)$ is one of the following types:

a constant, a polynomial, an exponential function on the form $e^{\alpha x}$, $\cos \beta x$ $\sin \beta x$, or finite sums and products of these types, then y_p has the same form as $g(x)$, but with general unknown coefficients to be determined.

The following table demonstrates the form of y_p depending upon the type of $g(x)$ incase of L.D.Es with constant coefficients.

$g(x)$	Form of y_p
3	A
x	$Ax+B$
$5x-9$	$Ax+B$
$2x^2+1$	Ax^2+Bx+C
x^3-2x	Ax^3+Bx^2+Cx+D
$7e^{3x}$	Ae^{3x}
xe^{5x}	$(Ax+B)e^{5x}$
$(6x^2+x)e^{5x}$	$(Ax^2+Bx+C)e^{5x}$
$3\sin x$	$A\cos x+B\sin x$
$x\cos x$	$(Ax+B)\cos x+(Cx+D)\sin x$

$g(x)$	Form of y_p
$(x^2 - x + 3)\cos x$	$(Ax^2 + Bx + C)\cos x + (Dx^2 + Ex + F)\sin x$
$4e^x \cos x$	$Ae^x \cos x + Be^x \sin x$
$xe^{-7x} \sin x$	$(Ax + B)e^{-7x} \cos x + (Cx + D)e^{-7x} \sin x$
$3\sin x + 4\cos 3x$	$A\cos x + B\sin x + C\cos 3x + D\sin 3x$
$4\sin x - 9\cos x$	$A\cos x + B\sin x$

Example 1. Solve the following D. equation:

$$y'' - 5y' + 6y = 12. \qquad (1)$$

Solution. The associated Hom. E. is $y'' - 5y' + 6y = 0$, hence the aux. eq. is

$$m^2 - 5m + 6 = 0 \Rightarrow (m-3)(m-2) = 0 \Rightarrow m = 3, 2.$$

Therefore $y_c = c_1 e^{3x} + c_2 e^{2x}$.

For y_p we have

$g(x)=12$ hence y_p is on the form $y_p = A$, where A is a constant to be determined.

But $y = A \Rightarrow y' = y'' = 0.$ Using these values in Eq.(1) we get $6A = 12 \Rightarrow A = 2 \Rightarrow y_p = 2,$

hence the general solution is

$$y = y_c + y_p$$

$$= c_1 e^{3x} + c_2 e^{2x} + 2.$$

Example 2. Solve the D. E.

$$y'' - 5y' + 6y = 4x. \qquad (1)$$

Solution. The associated Hom. E. is $y''-5y'+6y=0,$ hence from Example 1 we have $y_c=c_1e^{3x}+c_2e^{2x}.$

For y_p we have $g(x)=4x,$ hence y_p is on the form $y_p=Ax+B,$ where A and B are constants to be determined. But $y_p=Ax+B\Rightarrow y'=A, y''=0.$

Using these values in Eq.(1) implies

$$-5A+6(Ax+B)=2x$$

$$\Rightarrow 6Ax+(6B-5A)=2x. \qquad (2)$$

Comparing coefficients on both sides of Eq.(2) we get $6A=2,\ \ 6B-5A=0\Rightarrow A=\frac{1}{3},\ B=\frac{5}{18},$

hence $y_p=\frac{1}{3}x+\frac{5}{18},$ and the general solution is

$$y=y_c+y_p=c_1e^{3x}+c_2e^{2x}+\frac{1}{3}x+\frac{5}{18}.$$

Example 3. Solve $y''-3y'+2y=6e^{-x}$, (1)

Solution. The associated Hom. E. is $y''-3y'+2y=0$,

hence the aux. equation and it's roots are

$$m^2-3m+2=0 \Rightarrow (m-2)(m-1)=0 \Rightarrow m=2,\ 1,$$

therefore $y_c=c_1e^{2x}+c_2e^{x}$.

For y_p we have $g(x)=6e^{-x}$, hence y_p is on the form

$y_p=Ae^{-x}$, where A is a constants to be determined. But $y_p=Ae^{-x} \Rightarrow y'=-Ae^{-x},\ y''=Ae^{-x}$.

Using these values in Eq.(1) we get $6A=6 \Rightarrow A=1$,

therefore $y_p=e^{-x}$, and the general solution is

$$y=y_c+y_p=c_1e^{2x}+c_2e^{x}+e^{-x}.$$

Example 4. Solve $y''-3y'+2y=2-e^{3x}$.   (1)

Solution. The associated Hom. E. is $y''-3y'+2y=0$, hence from Example 3 we have $y_c=c_1e^{2x}+c_2e^{x}$.

For y_p we have $g(x)=g_1(x)+g_2(x)$, where

$$g_1(x)=2 \quad \Rightarrow y_{p_1}=A$$

$$g_2(x)=-e^{-x} \Rightarrow y_{p_2}=Be^{-x},$$

Hence y_p is on the form $y_p=y_{p_1}+y_{p_2}=A+Be^{-x}$,

Which implies $y'=-Be^{-x},\ y''=Be^{-x}$.

Using these values in Eq.(1) we get

$$A+6Be^{-x}=2-e^{-x} \quad \Rightarrow A=2,\ B=\tfrac{-1}{6} \Rightarrow y_p=2-\tfrac{1}{6}e^{-x}.$$

Therefore the general solution is $y=c_1e^{2x}+c_2e^{x}+2-\frac{1}{6}e^{-x}$.

Example 5. Solve $y''-3y'+2y = 2x - 3\sin x. \qquad (1)$

Solution. The associated Hom. E. is $y''-3y'+2y = 0,$

hence from Example 3 we have $y_c = c_1 e^{2x} + c_2 e^{x}.$

For y_p we have $g(x) = g_1(x) + g_2(x),$ where

$$g_1(x) = 2x \quad \Rightarrow y_{p_1} = Ax + B$$

$$g_2(x) = -3\sin x \Rightarrow y_{p_2} = C\cos x + D\sin x.$$

Hence y_p is on the form

$$y_p = y_{p_1} + y_{p_2} = Ax + B + C\cos x + D\sin x,$$

which implies

$$y'_p = A - C\sin x + D\cos x, \; y'_p = -C\cos x - D\sin x.$$

Using these values in Eq.(1) we obtain

$$2Ax + 2B - 3A + (C - 3D)\cos x + (D + 3C)\sin x = 2x - 3\sin x,$$

which implies $2A = 2,\ 2B - 3A = 0,\ C - 3D = 0,\ D + 3C = -3,$

hence $A = 1,\ B = \frac{3}{2},\ C = \frac{-9}{10},\ D = \frac{-3}{10}.$

Therefore $y_p = x + \frac{3}{2} - \frac{9}{10}\cos x - \frac{3}{10}\sin x,$ and the general solution is

$$y_p = y_c + y_p = c_1 e^{2x} + c_2 e^{x} + x + \frac{3}{2} - \frac{9}{10}\cos x - \frac{3}{10}\sin x.$$

Example 6. $y'' - 3y' + 2y = (3x - 2)e^{-x}.$ (1)

Solution. The associated Hom. E. is $y'' - 3y' + 2y = 0,$

hence from Example 3 we have $y_c = c_1 e^{2x} + c_2 e^{x}.$

For y_p we have $g(x) = (3x - 2)e^{-x}$, therefore y_p is on the form $y_p = (Ax + B)e^{-x},$ which implies

$$y'_p = Ae^{-x} - (Ax+B)e^{-x}, \quad y''_p = -2Ae^{-x} + (Ax+B)e^{-x}.$$

Using these values in Eq.(1) we get

$$6Ax - 5A + 6B = 3x - 2 \Rightarrow A = \tfrac{1}{2},\ B = \tfrac{5}{12} \Rightarrow y_p = \tfrac{1}{2}x + \tfrac{5}{12},$$

hence the general solution is

$$y_p = y_c + y_p = c_1 e^{2x} + c_2 e^{x} + \tfrac{1}{2}x + \tfrac{5}{12}.$$

Remark.

Assume that the particular solution of a nonhom. L.D.E. is on the form

$$y_p = y_{p_1} + \dots + y_{p_k}.$$

If there is a term in y_{p_i} duplicates a term in y_c, then this y_{p_i} must be multiplied by x^s, where s is the smallest positive integer that eliminates the duplication. In fact s is the multiplicity of the root of the associated auxiliary equation which causes the duplication.

Example 7. Solve $y''-2y'+y = x+4e^{x}$. (1)

Solution. The associated Hom. E. is $y''-2y'+y = 0$,

hence the aux. equation and it's roots are

$$m^2 - 2m + 1 = 0 \Rightarrow (m-1)(m-1) = 0 \Rightarrow m = 1, 1,$$

therefore $y_c = c_1 e^x + c_2 x e^x$.

For y_p we have $g(x) = g_1(x) + g_2(x)$, where

$$g_1(x) = 2x \quad \Rightarrow y_{p_1} = Ax + B,$$

$$g_2(x) = 4e^x \Rightarrow y_{p_2} = Ce^x.$$

It is clear that the term in y_{p_2} duplicates a term in y_c,

thus y_{p_2} must be multiplied by x^2 to eliminate this duplication. Hence $y_p = y_{p_1} + x^2 y_{p_2} = Ax + B + Cx^2 e^x$, Which implies

$$y'_p = A + 2Cxe^x + Cx^2e^x, \;\; y''_p = 2Ce^x + 4Cxe^x + Cx^2e^x$$

Using these values in Eq.(1) we get

$$Ax + B - 2A + 2Ce^x = x + 4e^x$$

$$\Rightarrow A = 1, \; B - 2A = 0, \;\; 2C = 4 \;\; \Rightarrow B = 2, \, C = 2,$$

therefore $y_p = x + 2 + 2x^2e^x$, and the he general solution is $y = c_1e^{2x} + c_2e^x + x + 2 + 2x^2e^x$.

Example 8. Find the form of the particular solution for each of the following differential equations

(1) $y^{(5)} - y''' = x + 2 - 3e^x + 5x\cos x.$

Solution. The auxiliary equation is

Which implies

$$y'_p = A + 2Cxe^x + Cx^2e^x, \quad y''_p = 2Ce^x + 4Cxe^x + Cx^2e^x$$

Using these values in Eq.(1) we get

$$Ax + B - 2A + 2Ce^x = x + 4e^x$$

$$\Rightarrow A = 1, \; B - 2A = 0, \; 2C = 4 \;\; \Rightarrow B = 2, \; C = 2,$$

therefore $y_p = x + 2 + 2x^2e^x$, and the he general solution is $y = c_1e^{2x} + c_2e^x + x + 2 + 2x^2e^x.$

Example 8. Find the form of the particular solution of the following differential equation

$$y^{(5)} - y''' = x + 2 - 3e^x + 5x\cos x.$$

Solution. The auxiliary equation is

$$m^5 - m^3 = 0 \Rightarrow m = 0, 0, 0, 1, -1.$$

Hence $y_c = c_1 + c_2 x + c_3 x^2 + c_4 e^x + c_5 e^{-x}.$

Now $g(x) = g_1(x) + g_2(x) + g_3(x)$, where

$$g_1(x) = 7x + 2 \quad \Rightarrow y_{p_1} = Ax + B,$$

$$g_2(x) = -3e^x \quad \Rightarrow y_{p_2} = Ce^x,$$

$$g_3(x) = 5x\cos x \Rightarrow y_{p_3} = (Dx + E)\cos x + (Fx + G)\sin x.$$

It is clear that there are terms in y_{p_1} duplicate terms in y_c therefore y_{p_1} must be multiplied by x^3 to eliminate this duplication. Also, the term in y_{p_2} duplicate a term in y_c, therefore y_{p_2} must be multiplied by x. Hence y_p is on the form

$$y_p = x^3(Ax + B) + Cxe^x + (Dx + E)\cos x + (Fx + G)\sin x.$$

Example 8. Find the form of the particular solution of the following differential equation

$$y^{(6)} + 2y^{(4)} + y'' = x^2 - 5e^{3x} - \cos x + 7\sin 3x.$$

Solution. The auxiliary equation is

$$m^6 + 2m^4 + m^2 = 0 \Rightarrow m^2(m^2+1)^2 = 0 \Rightarrow m = 0, 0, \pm i, \pm i,$$

hence $y_c = c_1 + c_2 x + c_3 \cos x + c_4 \sin x + c_5 x\cos x + c_6 x\sin x.$

Now $g(x) = g_1(x) + g_2(x) + g_3(x) + g_4(x)$ where

$$g_1(x) = x^2 \quad \Rightarrow y_{p_1} = Ax^2 + Bx + C,$$

$$g_2(x) = -5e^{3x} \quad \Rightarrow y_{p_2} = De^{3x},$$

$$g_3(x) = -\cos x \Rightarrow y_{p_3} = E\cos x + F\sin x,$$

$$g_4(x) = 7\sin 3x \Rightarrow y_{p_3} = G\cos 3x + H\sin 3x.$$

It is clear that there are terms in y_{p1} duplicate terms in y_c therefore y_{p1} must be multiplied by x^2 to eliminate this duplication. Also, there are terms in y_{p_3} duplicate terms in y_c, therefore y_{p_3} must be multiplied by x. Hence y_p is on the form

$$y_p = x^2(Ax^2 + Bx + C) + Ce^{3x} + x(E\cos x + F\sin x) + Gx\cos 3x + Hx\sin x.$$