# Cauchy-Euler Equation

An nth order linear DE

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1 x \frac{dy}{dx} + a_0 y = g(x),$$

where $a_n, a_{n-1}, \dots, a_0$ are constants, is called Cauchy-Euler equation.

Example: $(i)\ 3x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + 5y = 0$

$$(ii)\ x^4 \frac{d^4 y}{dx^4} - x^2 \frac{d^2 y}{d^2 x} - 3y = \ln x$$

We shall confine our attention to finding the general solution of Cauchy-Euler equation on the interval $(0, \infty)$.

# Method of solution

First consider a first order homogeneous Cauchy-Euler Equation:

$$ax\frac{dy}{dx} + by = 0.$$

It is easy to see that the solution is given by

$$y = cx^m, \; where \; m = \frac{-b}{a}.$$

Now, consider a second order equation

$$ax^2\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad (1)$$

and suppose that $y = x^m$ is a solution of (1), where $m$ is a constant to be determined.

$$\Rightarrow y' = mx^{m-1},\ y'' = m(m-1)x^{m-2}$$

Using the values of $y, y', y''$ in (1) we obtain

$$x^m\left[am(m-1) + bm + c\right] = 0$$

But $x^m \neq 0$, therefore

$$am(m-1) + bm + c = 0$$

Or

$$am^2 + (b-a)m + c = 0. \quad (2)$$

Thus, $y = x^m$ is a solution of (1) whenever $m$ is a root of the the auxiliary equation (2).

In solving Eq.(2) we have three cases:

**Case1:**

Equation 2 has two distinct real roots, say $m_1, m_2$, then $y_1 = x^{m_1},\ y_2 = x^{m_2}$ are two linearly independent solutions of Eq.(1), and hence the general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 x^{m_1} + c_2 x^{m_2},\ c_1, c_2 \in R.$$

Example:

Solve the DE $\quad x^2 y'' - 2xy' - 4y = 0. \qquad (1)$

Let $y = x^m$, then $y' = mx^{m-1},\ y'' = m(m-1)x^{m-2}$

Using these values in (1) we obtain

$$x^m\left[m(m-1) - 2m - 4\right] = 0$$

or

$$x^m\left[m^2 - 3m - 4\right] = 0$$

But $x^m \neq 0 \Rightarrow m^2 - 3m - 4 = 0$

$$\Rightarrow (m-4)(m+1) = 0$$

$$\Rightarrow m = 4, -1$$

Hence the general solution is

$$y = c_1 x^4 + c_2 x^{-1},\ c_1, c_2 \in R.$$

**Case 2:**

Equation 2 has two repeated real roots, say $m_1 = m_2 = \lambda$, then $y_1 = x^{\lambda},\ y_2 = x^{\lambda} \ln x$ are two linearly independent solutions of Eq.(1), and hence the general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 x^{\lambda} + c_2 x^{\lambda} \ln x, \; c_1, c_2 \in R.$$

Example:

Solve the DE $\quad 4x^2 y'' + 8xy' + y = 0. \qquad (1)$

Let $y = x^m$, then $y' = mx^{m-1}, \; y'' = m(m-1)x^{m-2}$

Using these values in (1) we obtain

$$x^m \left[4m(m-1) + 8m + 1\right] = 0$$

or $x^m \left[4m^2 + 4m + 1\right] = 0$

But $x^m \neq 0 \Rightarrow 4m^2 + 4m + 1 = 0$

$$\Rightarrow (2m+1)(2m+1) = 0$$

$$\Rightarrow m = \tfrac{-1}{2}, \tfrac{-1}{2}$$

Therefore the general solution is $y = c_1 x^{\frac{-1}{2}} + c_2 x^{\frac{-1}{2}} \ln x.$

**Case 3:**

Equation 2 has two complex conjugate roots, say

$m_1 = \alpha + \beta i,\ \ m_2 = \alpha - \beta i$, then

$y_1 = x^{\alpha + i\beta},\ y_2 = x^{\alpha - i\beta}$ are two linearly independent solutions of (1).

However, using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$

the two independent solutions can be reformulated in the form $y_1 = x^{\alpha}\cos(\beta \ln x),\ y_2 = x^{\alpha}\sin(\beta \ln x),$

and hence the general solution of Eq.(1) is

$$y = x^{\alpha}\left[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)\right],\ \ c_1, c_2 \in R.$$

## Example:

Solve the DE $x^2 y'' + 3xy' + 3y = 0.$ (1)

Let $y = x^m$, then $y' = mx^{m-1},\ y'' = m(m-1)x^{m-2}$

Using these values in (1) we obtain

$$x^m\left[m(m-1)+3m+3\right]=0,$$

or $x^m\left[m^2+2m+3\right]=0.$

But $x^m \neq 0 \Rightarrow m^2+2m+3=0,$

$$\Rightarrow m=-1\pm\sqrt{2}i,$$

$$\Rightarrow \alpha=-1,\ \beta=\sqrt{2}.$$

Therefore the general solution is

$$y=x^{-1}\left[c_1\cos(\sqrt{2}\ln x)+c_2\sin(\sqrt{2}\ln x)\right]$$

## Example:

Solve the DE $y''' - \frac{6}{x^3}y = 0.$ (1)

Multiplying both sides by $x^3$, we obtain

$$x^3 y''' - 6y = 0. \qquad (2)$$

Now, let $y = x^m$, then $y''' = m(m-1)(m-2)x^{m-3}$

Using these values in (2) we get

$$x^m\left[m(m-1)(m-2)-6\right] = 0.$$

or

$$x^m\left[m^3 - 3m^2 + 2m - 6\right] = 0,$$

But $x^m \neq 0 \Rightarrow m^3 + 3m^2 + 2m - 6 = 0,$

$$\Rightarrow m = 3, \pm\sqrt{2}i$$

$$y = x^{-1}\left[c_1\cos(\sqrt{2}\ln x) + c_2\sin(\sqrt{2}\ln x)\right]$$

Therefore the independent solutions are

$$y_1 = x^3,\ y_2 = \cos(\sqrt{2}\ln x),\ y_3 = \sin(\sqrt{2}\ln x)$$

And the general solution is

$$y = c_1 x^3 + c_2\cos(\sqrt{2}\ln x) + c_3\sin(\sqrt{2}\ln x).$$

Example

Solve the DE $\quad xy'' - y' + \frac{1}{x}y = 2. \qquad (1)$

Multiplying both sides by $x$, we obtain

$$x^2 y'' - xy' + y = 2x. \qquad (2)$$

This is a nonhomogeneous Cauchy-Euler equation,

therefore the general solution is of the form $y = y_c + y_p$.

For $y_c$ let $y = x^m$, then $y' = mx^{m-1},\ y'' = m(m-1)x^{m-2}$

Using these values in $x^2 y'' - xy' + y = 0$

Imply $\quad x^m\left[m(m-1) - m + 1\right] = 0,$

or $\quad x^m\left[m^2 - 2m + 1\right] = 0.$

But $\; x^m \neq 0 \Rightarrow m^2 - 2m + 1 = 0,$

$$\Rightarrow m = 1, 1.$$

Hence, the independent solutions are $y_1 = x,\; y_2 = x\ln x,$

and $\; y_c = c_1 x + c_2 x \ln x.$

For $y_p$ , we apply the variation of parameter.

$$w = \begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix} = x,\; w_1 = \begin{vmatrix} 0 & x\ln x \\ \frac{2}{x} & 1+\ln x \end{vmatrix} = -2\ln x,\; w_2 = \begin{vmatrix} x & 0 \\ 1 & \frac{2}{x} \end{vmatrix} = 2.$$

Hence, $u_1 = \int \frac{w_1}{w}\, dx = -(\ln x)^2$

$$u_2 = \int \frac{w_2}{w}\, dx = 2(\ln x)$$

$$\Rightarrow y_p = u_1 y_1 + u_2 y_2$$

$$= -x(\ln x)^2 + 2x(\ln x)^2$$

$$= x(\ln x)^2.$$

Therefore the general solution is

$$y = y_c + y_p$$

$$= c_1 x + c_2 x \ln x + x(\ln x)2.$$

Cauchy-Euler equation can be reduced to a linear D.E. with constant coefficients using the substitution

$$x = e^t \quad or \quad t = \ln x.$$

Example

Use the substitution $x = e^t$ $or$ $t = \ln x$ to solve the D.E.

$$x^2 y'' - 3xy' + 3y = 0. \qquad (1)$$

Solution. By the chain rule we have

$$y' = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt},$$

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right).$$

Using these values in Eq.(1) we get

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = 0, \qquad (2)$$

Which is homogeneous L.D.E. with constant coefficients.

Hence the auxiliary equation is

$$m^2 - 4m + 3 = 0,$$

$$or \ \ (m-1)(m-3) = 0 \Rightarrow m = 1, \ or \ \ m = 3.$$

Hence the solution of Eq.(2) is

$$y = c_1 e^{t} + c_2 e^{3t}.$$

Therefore the solution of Eq.(1) is given by

$$y = c_1 e^{\ln x} + c_2 e^{3 \ln x}$$

$$= c_1 x + c_2 x^3.$$

# General form of Cauchy-Euler Equation

The general form of Cauchy-Euler equation is

$$a_n(\alpha x+\beta)^n \frac{d^n y}{dx^n} + a_{n-1}(\alpha x+\beta)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(\alpha x+\beta)\frac{dy}{dx} + a_0(\alpha x+\beta)y = g(x),$$

where $a_n, a_{n-1}, \dots, a_0, \alpha, \beta, \ \beta \neq 0$ are constants.

Example. Solve the D.E.

$$(2x-1)^2 y'' - (2x-1)y' - 4y = 0. \quad (1)$$

Solution. Let

$$y = (2x-1)^m \Rightarrow y' = 2m(2x-1)^{m-1}, \ y'' = 4m(m-1)(2x-1)^{m-2}.$$

Using these values in Eq.(1) we obtain

$$(2x-1)^m[4m(m-1) - 2m - 4] = 0$$

$$\Rightarrow 2m^2 - 3m - 2 = 0 \Rightarrow m = -\tfrac{1}{2} \ or \ m = 2.$$

Hence the general solution is

$$y = c_1(2x-1)^2 + c_2(2x-1)^{\frac{-1}{2}}.$$

Homework

Solve the D.E.

$$(3x+2)^2 y'' + 10(3x+2)y' + 9y = 0.$$