Homogeneous Linear D.Es with constant coefficients

A general n^{th} order H.L.D.E. with constant coefficients is on the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_0 y = 0, \qquad (1)$$

where $a_0, a_1, \ldots, a_n$ are constants.

It is easy to see that $y = e^{mx}$ is a solution of the D.E.

$a\,y' + by = 0$ where m is given by $m = \frac{-b}{a}$.

Does higher order equations have solutions on this form?

If yes, what are the values of m?.

Suppose $y = e^{mx}$ is a solution of the second order D.E.

$$ay'' + by' + cy = 0. \qquad (2)$$

Since $y = e^{mx} \Rightarrow y' = me^{mx}, \; y'' = m^2 e^{mx},$

using these values in Eq.(2) we get

$$e^{mx}(am^2 + bm + c) = 0$$

$$\Rightarrow am^2 + bm + c = 0.$$

Therefore $y = e^{mx}$ is a solution of Eq.(2) if and only if m is a root of the algebraic equation

$$am^2 + bm + c = 0. \qquad (3)$$

Equation (3) is called the auxiliary (or the characteristic) equation of the D.E. (2).

In solving the auxiliary Eq.(3) for m we have three cases:

Case 1. The two roots are real and different, say $m = m_1,\ m = m_2,$ then the fundamental solutions are $y_1 = e^{m_1 x}$ *and* $y_2 = e^{m_2 x},$ hence the general solution is given by $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$

Example. Solve the D.E. $y'' - 7y' + 12y = 0.$

Solution. The auxiliary equation is

$$m^2 - 7m + 12 = 0 \Rightarrow (m-3)(m-4) = 0 \Rightarrow m = 3, m = 4.$$

Hence the general solution is $y = c_1 e^{3x} + c_2 e^{4x}.$

Case 2. The two roots are real and equal, say

$m=\lambda, m=\lambda,$ then the fundamental solutions are

$y_1 = e^{\lambda x}$ and $y_2 = xe^{\lambda x},$ hence the general solution is given by $y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}.$

Example. Solve the D.E. $y'' - 10y' + 25y = 0.$

Solution. The auxiliary equation is

$$m^2 - 10m + 25 = 0 \Rightarrow (m-5)(m-5) = 0 \Rightarrow m = 5, m = 5.$$

Hence the general solution is $y = c_1 e^{5x} + c_2 x e^{5x}.$

Case 3. The two roots are complex conjugates, say

$m_1 = \alpha + \beta i,\ m_2 = \alpha - \beta i,$

Hence we get the two solutions

$$y_1 = e^{\alpha+\beta i} \quad and \quad y_2 = e^{\alpha-\beta i},$$

and using Euler's formula the fundamental solutions may be written as

$$y_1 = e^{\alpha x}\cos(\beta x) \; and \quad y_2 = e^{\alpha x}\sin(\beta x),$$

hence the general solution is

$$y = e^{\alpha x}[c_1 \cos(\beta x) + c_2 \sin(\beta x)].$$

Example. Solve the D.E. $y''+2y'+5y = 0.$

Solution. The auxiliary equation is

$$m^2 + 2m + 5 = 0 \Rightarrow m = -1 \pm 2i,$$

Hence we get the two solutions

$$y = e^{-x}[c_1 \cos(2x) + c_2 \sin(2x)].$$

Example. Solve the D.E. $y''' - 8y = 0.$

Solution. The auxiliary equation is

$$m^3 - 8 = 0 \Rightarrow (m-2)(m^2 + 2m + 4) = 0$$

$$\Rightarrow m = 2 \;\; or \;\; m^2 + 2m + 4 = 0,$$

$$\Rightarrow m = 2 \;\; or \;\; m = -1 \pm \sqrt{3}i,$$

hence the general solution is

$$y = c_1 e^{2x} + e^{-x}[c_2 \cos(\sqrt{3}x) + c_3 \sin(\sqrt{3}x)].$$

Example. Solve the D.E. $y'''-2y''-4y'+8y=0.$

Solution. The auxiliary equation is

$$m^3-2m^2-4m+8=0,$$

$$\Rightarrow m^2(m-2)-4(m-2)=0,$$

$$\Rightarrow (m-2)(m^2-4)=0,$$

$$\Rightarrow (m-2)(m-2)(m+2)=0,$$

$$\Rightarrow m=2,2,-2,$$

hence the general solution is $y=c_1e^{2x}+c_2xe^{2x}+c_3e^{-2x}.$

Example. Solve the D.E. $y'''-3y'-2y=0.$

Solution. The auxiliary equation is

$$m^3-3m-2=0 \Rightarrow (m-2)(m^2-1)=0,$$

$$\Rightarrow (m-2)(m-1)(m+1)=0,$$

$$\Rightarrow m=2,1,-1,$$

hence the general solution is

$$y = c_1 e^{2x} + c_2 e^{x} + c_3 e^{-x}.$$

Example. Solve the initial value problem

$$\begin{cases} y'' - 2y' - 3y = 0, \\ y(0) = 3,\ y'(0) = 1. \end{cases}$$

Solution. The auxiliary equation is

$$m^3 - 2m - 3 = 0 \Rightarrow (m-3)(m+1) = 0,$$

$$\Rightarrow m = 3, -1,$$

hence the general solution is $y = c_1 e^{3x} + c_2 e^{-x}.$

But $y(0) = 3 \Rightarrow c_1 + c_2 = 3,$ and $y'(0) = 1 \Rightarrow 3c_1 - c_2 = 1,$

Hence $c_1 = 1, c_2 = 2$ and the solution of the i.v.p. is

$$y = e^{3x} + 2e^{-x}.$$

Example. Find a H.L.D.E. with constant coefficients if the roots of the corresponding auxiliary equation are $m=1, m=-1, m=2-3i.$

Solution. The factors of the auxiliary equation are

$$(m-1),\ (m+1),(m-2)^2-(3i)^2=m^2-4m+13,$$

hence the auxiliary equation is

$$(m-1)\ (m+1)(m^2-4m+13)=0,$$

$$or\ \ m^4-4m^3+12m^2+4m-13=0.$$

Therefore the D. E. is

$$y^{(4)}-4y'''+12y''+4y'-13y=0.$$

Example. Find a H.L.D.E. with constant coefficients which has the following set of solutions

$$3x, -7e^{-2x}, e^{x}.$$

Solution. It follows that the roots of the auxiliary equation are $m = 0, m = 0, m = -2, m = 1,$

hence the auxiliary equation is

$$m^2 (m+2)(m-1) = 0,$$

$$or\ m^4 + m^3 - 2m^2 = 0.$$

Therefore the D. E. is

$$y^{(4)} + y''' - 2y'' = 0.$$

Example. Find a H.L.D.E. with constant coefficients which has the following set of solutions

$$5,\ e^{2x}\cos 3x,\ -\sin x.$$

Solution. It follows that the roots of the auxiliary equation are $m = 0, m = 2 \pm 3i, m = i,$

hence the auxiliary equation is

$$m[(m-2)^2 + 9](m^2+1) = 0,$$

$$\Rightarrow m(m^2 - 4m + 13)(m^2+1)$$

$$or \quad m^5 - 4m^4 + 14m^3 - 4m^2 + 13m = 0,$$

Therefore the D. E. is

$$y^{(5)} - 4y^{(4)} + 14y''' - 4y'' + 13y = 0.$$