# Constructing a second solution from a known one. (Reduction of order)

Consider a second order L.D.E. on the standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0, \qquad (1)$$

Where $P$ and $Q$ are continuous on some interval $I$.

Let $y_1(x)$ be a given solution of Eq.(1) defined on $I$ and $y_1(x) \neq 0$ for all $x$ in $I$.

Suppose $y = uy_1$ is a solution of Eq.(1). Then,

$$y' = u'y_1 + uy'_1$$

$$y'' = u''y_1 + 2u'y'_1 + uy''_1.$$

Substituting these values in Eq.(1) we get

$$y_1 u'' + (2y'_1 + Py_1)u' = 0.$$

Let $W = u' \Rightarrow W' = u''$, using these values in Eq.(2) we get

$$\frac{dW}{W} + (2\frac{y'_1}{y_1} + P)dx = 0.$$

Which is separable first order D.E., hence we have

$$\ln|W| + 2\ln|y_1| + \int P(x)dx + c = 0$$

$$or \quad Wy^2{}_1 = c_1 e^{-\int P(x)dx} \Rightarrow W = c_1 \frac{e^{-\int P(x)dx}}{y^2{}_1}$$

Hence $u = c_1 \int \frac{e^{-\int P(x)dx}}{y^2{}_1} dx + c_2,$

$$\Rightarrow y = c_1 y_1 \int \frac{e^{-\int P(x)dx}}{y^2{}_1} dx + c_2 y_1.$$

Therefore the second solution is given by

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y^2{}_1}\,dx.$$

It is easy to see that $y_1$ *and* $y_2$ are linearly independent on the interval $I$.

Example 1. Use reduction of order to solve the D.E.

$$xy'' - (x+1)y' + y = 0,\ x > 0, \qquad \textbf{(1)}$$

if $y_1 = e^x$ is a given solution.

Solution. Let $y = uy_1 = ue^x$

$$\Rightarrow y' = u'e^x + ue^x,\ y'' = u''e^x + 2u'e^x + ue^x.$$

Using these values in Eq.(1) we obtain

$$xe^{x}u''+(x-1)e^{x}u'=0, \quad \div xe^{x} \Rightarrow u''+(1-\tfrac{1}{x})u'=0.$$

Let $W=u' \Rightarrow \frac{dW}{dx}=u''$,

hence we have $\frac{dW}{dx}+(1-\frac{1}{x})W=0$,

which is a first order L.D.E as well as separable D.E.

Separating variables we get

$$\frac{dW}{W}=(\tfrac{1}{x}-1)dx$$

$$\Rightarrow \ln|W|=\ln(x)-x+c$$

$$or \quad W=c_1 xe^{-x}, \; c_1=e^{c}.$$

Hence $u=\int c_1 xe^{-x}\,dx$

$$=c_1(-xe^{-x}-e^{-x})+c_2.$$

Therefore the solution of the D.E. is

$$y=y_1 u \quad =e^{x}\{c_1(-xe^{-x}-e^{-x})+c_2\}=c_1(-x-1)+c_2 e^{x}.$$

Example 2. Solve the D.E.

$$x^2 y'' - xy' + (x^2 - \tfrac{1}{4})y = 0, \; x \in (0, \pi) \qquad (1)$$

given that $y_1 = \frac{\cos x}{\sqrt{x}}$ is a solution.

Solution. Put the D.E on the standard form

$$y'' + P(x)y' + Q(x)y = 0,$$

then apply the formula $y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y^2{}_1}\,dx.$

Thus we have $y'' - \frac{1}{x}y' + (1 - \frac{1}{4x^2})y = 0,$

hence $y_2 = y_1 \int \frac{e^{-\int \frac{1}{x}dx}}{\frac{\cos^2 x}{x}}\,dx = y_1 \int \frac{e^{-\ln x}}{\frac{\cos^2 x}{x}}\,dx = y_1 \int \frac{1}{\cos^2 x}\,dx$

$$= y_1 \int \sec^2 x\,dx = \frac{\sin x}{\sqrt{x}},$$

and the general solution is

$$y = c_1 \frac{\cos x}{\sqrt{x}} + c_2 \frac{\sin x}{\sqrt{x}}.$$

Example 3. Use reduction of to solve the D.E.

$$y''-y'-2y = x, \qquad (1)$$

Given that $y_1 = e^{-x}$ is a solution.

Solution. Let $y = uy_1 = ue^{-x}$

$$\Rightarrow y' = u'e^{-x} - ue^{-x}, \; y'' = u''e^{-x} - 2u'e^{-x} - ue^{-x}$$

Using these values in Eq.(1) we have

$$u''-3u' = xe^{x}.$$

Let $W = u' \Rightarrow \frac{dW}{dx} = u''$, which implies

$$\frac{dW}{dx} - 3W = xe^{x},$$

Which is first order L.D.E. with an integrating factor

$$\mu(x) = e^{-3x}.$$

Hence we have

$$e^{-3x}W = c_1 + \int xe^{-2x}dx$$

$$= c_1 + -\tfrac{1}{2}xe^{-2x} - \tfrac{1}{4}e^{-2x}$$

$$\Rightarrow W = c_1 e^{3x} - (\tfrac{1}{2}x + \tfrac{1}{4})e^{x}$$

$$\Rightarrow u = \int c_1 e^{3x} - (\tfrac{1}{2}x + \tfrac{1}{4})e^{x}\, dx$$

$$= \tfrac{1}{3}c_1 e^{3x} - (\tfrac{1}{2}x - \tfrac{1}{4})e^{x} + c_2$$

Hence the general solution is

$$y = y_1 u \;\; = e^{-x}\{\tfrac{1}{3}c_1 e^{3x} - (\tfrac{1}{2}x - \tfrac{1}{4})e^{x} + c_2\}$$

$$= \tfrac{1}{3}c_1 e^{2x} - (\tfrac{1}{2}x - \tfrac{1}{4}) + c_2 e^{-x}.$$

Notice that expression $-(\tfrac{1}{2}x - \tfrac{1}{4})$ represents the particular solution since the D.E. is nonhomogeneous.

HW. Solve the D.E.

$$x^2 y'' - xy' + 2y = 0,\ x \in (0, \infty),$$

if $y_1 = x\cos(\ln x)$ is a given solution.