Chapter 4: Interval Estimation

4.1 Let the observed value of the mean X of a random sample of size 20 from a distribution that is $N(\mu, 80)$ be 81.2. Find a 95 percent confidence interval for μ

Given $\boldsymbol{X_1}, \boldsymbol{X_2}, \ldots, \boldsymbol{X_{20}}$ $\sim$Normal$(\mu, 80)$ $\sigma^2 = 80 \Rightarrow \sigma = \sqrt{80}$

$\bar{X} = 81.2$

Since X_i~Normal and σ known $\Rightarrow \mu \in \bar{X} \pm Z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$

$\alpha = 0.05 \Rightarrow \frac{\alpha}{2} = 0.025 \Rightarrow 1 - \frac{\alpha}{2} = 0.975$

$\Rightarrow Z_{1-\frac{\alpha}{2}} = Z_{0.975} = 1.96$

$\Rightarrow \mu \in (81.2 \pm 1.96 \frac{\sqrt{80}}{\sqrt{20}})$,

$\Rightarrow \mu \in (77.28 , 85.12)$

4.2 Let X be the mean of a random sample of size n from a distribution that is $N(\mu, 9)$. Find n such that $P(\bar{X} - 1 < \mu < \bar{X} + 1) = 0.90$, approximately.

$\boldsymbol{X_1}, \boldsymbol{X_2}, \ldots, \boldsymbol{X_n}$ $\sim$Normal$(\mu, 9)$ $\sigma^2 = 9 \Rightarrow \sigma = 3$

$P(\bar{X} - 1 < \mu < \bar{X} + 1) = 0.90$

Since X_i~Normal and σ known $\Rightarrow \mu \in \bar{X} \pm Z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$

$1 - \alpha = 0.90 \Rightarrow \alpha = 0.10 \Rightarrow Z_{1-\frac{\alpha}{2}} = Z_{0.95} = 1.645$

$$\bar{X} - Z_{0.95} \frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + Z_{0.95} \frac{\sigma}{\sqrt{n}}$$

$$\bar{X} - 1 < \mu < \bar{X} + 1$$

$\bar{X} - Z_{0.95} \frac{\sigma}{\sqrt{n}} = \bar{X} - 1$ and $\bar{X} + Z_{0.95} \frac{\sigma}{\sqrt{n}} = \bar{X} + 1$

$$\Rightarrow Z_{0.95} \frac{\sigma}{\sqrt{n}} = 1$$

$\Rightarrow 1.645 \frac{3}{\sqrt{n}} = 1 \Rightarrow \frac{4.935}{\sqrt{n}} = 1 \Rightarrow n = (4.935)^2 = 24.35 \approx 25$

4.3 Let a random sample of size 17 from the normal distribution $N(\mu, \sigma^2)$ yield $\bar{X}$= 4.7 and S^2 = 5.76. Determine a 90% confidence interval for μ

Given $\boldsymbol{X_1}, \boldsymbol{X_2}, \dots, \boldsymbol{X_{17}}$ $\sim$Normal(μ, σ^2)

$\bar{X} = 4.7 \quad S^2 = 5.76 \quad$ n=17

Since $X_i \sim$Normal and σ unknown $n = 17 < 30\ (small) \Rightarrow \mu \in \bar{X} \pm t_{1-\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$

$\alpha = 0.10 \quad \Rightarrow 1 - \frac{\alpha}{2} = 0.95$

$\Rightarrow\ t_{1-\frac{\alpha}{2}, n-1} = t_{0.95,16} = 1.746$

$\Rightarrow \mu \in (4.7 \pm 1.741\ \frac{\sqrt{5.76}}{\sqrt{17}})$,

$\Rightarrow \mu \in (3.6836\ , 5.71632)$

4.4 If 8.6 7.9 8.3 8.4 6.4 8.4 9.8 7.2 7.8 7.5 are the observed values of a random sample of size 10 from a distribution that is $N(8, \sigma^2)$, construct a 90% confidence interval for σ^2.

Since $X_i \sim$Normal and μ known $\Rightarrow \sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma^2} \sim \chi^2{}_n$

$$\frac{\sum_{i=1}^{n}(X_i - \mu)^2}{\chi^2{}_{\frac{\alpha}{2}, n}} < \sigma^2 < \frac{\sum_{i=1}^{n}(X_i - \mu)^2}{\chi^2{}_{1-\frac{\alpha}{2}, n}}$$

$\alpha = 0.1 \quad \Rightarrow \frac{\alpha}{2} = 0.05 \ \Rightarrow 1 - \frac{\alpha}{2} = 0.95$

$\Rightarrow\ \chi^2{}_{0.95,10} = 3.94 \quad$ and $\quad \chi^2{}_{0.05,9} = 18.307$

$$\sum_{i=1}^{n}(X_i - 8)^2 = 7.51$$

$$\frac{\sum_{i=1}^{n}(X_i - 8)^2}{18.307} < \sigma^2 < \frac{\sum_{i=1}^{n}(X_i - 8)^2}{3.94}$$

$0.4102 < \sigma^2 < \ 1.9061$

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4.5 A random sample of size 15 from the normal distribution $N(\mu, \sigma^2)$ yields $\bar{X}$ = 3.2 and s^2 = 4.24. Determine a 90% confidence interval for $\sigma\sigma 2$

Since $X_i \sim$Normal and μ unknown $\Rightarrow \frac{(n-1)s^2}{\sigma^2} \sim \chi^2{}_{n-1}$

$$\Rightarrow \quad \frac{(n-1)s^2}{\chi^2{}_{\frac{\alpha}{2},n-1}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2{}_{1-\frac{\alpha}{2},n-1}}$$

$$1-\alpha = 0.90 \;\Rightarrow\; \alpha = 0.10 \Rightarrow \frac{\alpha}{2} = 0.05 \Rightarrow \; 1-\frac{\alpha}{2} = 0.95$$

$\chi^2{}_{\frac{\alpha}{2},n-1} = \chi^2{}_{0.05,14} = 23.6848$ and $\chi^2{}_{1-\frac{\alpha}{2},n-1} = \chi^2{}_{0.95,14} = 6.57063$

$$\Rightarrow \quad \frac{(14)(4.24)}{23.6848} < \sigma^2 < \frac{(14)(4.24)}{6.57063}$$

$$\Rightarrow \quad 2.5062 < \sigma^2 < 9.0341$$

4.6 Find a pivotal quantity based on a random sample of size n from $N(\theta, \theta^2)$ population, where $\theta > 0$. Use the pivotal quantity to set up a 1 - α confidence interval for θ .

Suppose that n<30 and since θ^2 unknown and since X ~Normal

We can use the pivotal quantity $\frac{\bar{X}-\theta}{S/\sqrt{n}} \sim t_{(n-1)}$

$$P\left(t_{\frac{\alpha}{2}} < \frac{\bar{X}-\theta}{S/\sqrt{n}} < t_{1-\frac{\alpha}{2}}\right) = 1 - \alpha$$

$$\Rightarrow \quad t_{\frac{\alpha}{2},n-1} < \frac{\bar{X}-\theta}{S/\sqrt{n}} < t_{1-\frac{\alpha}{2},n-1}$$

$$\Rightarrow \quad \frac{s}{\sqrt{n}} t_{\frac{\alpha}{2},n-1} < \bar{X} - \theta < \frac{s}{\sqrt{n}} t_{1-\frac{\alpha}{2},n-1} \quad \Rightarrow \quad \bar{X} + \frac{s}{\sqrt{n}} t_{\frac{\alpha}{2},n-1} < \theta < \bar{X} + \frac{s}{\sqrt{n}} t_{1-\frac{\alpha}{2},n-1}$$

Or

since X ~Normal and θ unknown

We can use the pivotal quantity $\frac{(n-1)s^2}{\theta^2} \sim \chi^2{}_{n-1}$

$$P\left(\chi^2{}_{1-\frac{\alpha}{2},n-1} < \frac{(n-1)s^2}{\theta^2} < \chi^2{}_{\frac{\alpha}{2},n-1}\right) = 1 - \alpha$$

$$\Rightarrow \quad \frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2},n-1}} < \theta^2 < \frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2},n-1}}$$

$$\Rightarrow \quad \sqrt{\frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2},n-1}}} < \theta < \sqrt{\frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2},n-1}}}$$

4.7 Let $\bar{X}$ denote the mean of a random sample of size 25 from a gamma distribution with 4 and $\beta > 0$. Use the central limit theorem to find an approximate 0.95 confidence interval for β

Given that $X \sim \text{Gamma}(4, \beta) \Rightarrow \mu = E(X) = 4\beta \quad V(X) = 4\beta^2$

By using the CLT $\Rightarrow \bar{X} \sim \text{Normal}\left(E(X), \frac{V(X)}{n}\right) \Rightarrow \bar{X} \sim \text{Normal}\left(4\beta, \frac{4\beta^2}{n}\right)$

Then, $\frac{\bar{X}-4\beta}{\sqrt{4\beta^2/n}} \sim \text{Normal}(0,1)$

$$\frac{\bar{X}-4\beta}{\sqrt{4\beta^2/25}} = \frac{\bar{X}-4\beta}{2\beta/5} = \frac{5\bar{X}-20\beta}{2\beta} = \frac{5\bar{X}}{2\beta} - 10 \sim \text{Normal}(0,1)$$

$$1-\alpha = 0.95 \Rightarrow \alpha = 0.05 \Rightarrow 1-\frac{\alpha}{2} = 0.975$$

$P\left(Z_{\frac{\alpha}{2}} < \frac{5\bar{X}}{2\beta} - 10 < Z_{1-\frac{\alpha}{2}}\right) = 0.95 \qquad Z_{1-\frac{\alpha}{2}} = Z_{0.975} = 1.96 \qquad , Z_{\frac{\alpha}{2}} = -Z_{1-\frac{\alpha}{2}}$

$$\Rightarrow \quad Z_{\frac{\alpha}{2}} < \frac{5\bar{X}}{2\beta} - 10 < Z_{1-\frac{\alpha}{2}}$$

$$\Rightarrow -1.96 < \frac{5\bar{X}}{2\beta} - 10 < 1.96$$

$$\Rightarrow 8.04 < \frac{5\bar{X}}{2\beta} < 11.96$$

$$\Rightarrow \frac{1}{11.96} < \frac{2\beta}{5\bar{X}} < \frac{1}{8.04}$$

$$\Rightarrow \frac{5\bar{X}}{(2)11.96} < \beta < \frac{5\bar{X}}{(2)8.04}$$

$\beta \in (0.21\bar{X}, 0.31\bar{X})$

4.8 Let $X_1, X_2, \ldots, X_6$ be a random sample of size 6 from a gamma distribution with parameters 1 and unknown $\beta > 0$. Discuss the construction of a 98% confidence interval for β

It is given that $X_i \sim \text{Gamma}(1, \beta)$ $\quad n = 6$ $\qquad M_{X_i}(t) = (1 - \beta t)^{-1}$

$\Rightarrow \sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \beta) \qquad \sum_{i=1}^{6} X_i \sim \text{Gamma}(6, \beta)$

$\Rightarrow \frac{2}{\beta} \sum_{i=1}^{n} X_i \sim \text{Gamma}(n, 2) \qquad \frac{2}{\beta} \sum_{i=1}^{6} X_i \sim \text{Gamma}(6,2)$

We know that $\chi^2{}_v \equiv \text{Gamma}\left(\alpha = \frac{v}{2}, \beta = 2\right)$

Check by using the M.G.F

$$M_{\frac{2}{\beta}\sum_{i=1}^{n} X_i}(t) = M_{\sum_{i=1}^{n} X_i}\left(\frac{2}{\beta} t\right) = E\left(e^{\sum_{i=1}^{n} X_i \left(\frac{2}{\beta} t\right)}\right)$$

$$= E\left(e^{\sum_{i=1}^{n} X_i \left(\frac{2}{\beta} t\right)}\right)$$

$$= E\left(e^{X_1 \left(\frac{2}{\beta} t\right)}\right) \ldots E\left(e^{X_n \left(\frac{2}{\beta} t\right)}\right)$$

$$= \left(1 - \beta \frac{2}{\beta} t\right)^{-1} \ldots \left(1 - \beta \frac{2}{\beta} t\right)^{-1}$$

$$= (1 - 2t)^{-n}$$

We know that $\chi^2{}_v \equiv \text{Gamma}\left(\alpha = \frac{v}{2}, \beta = 2\right)$

We have $\frac{2}{\beta} \sum_{i=1}^{6} X_i \sim \text{Gamma}(6,2)$

$\Rightarrow \frac{2}{\beta} \sum_{i=1}^{6} X_i \sim \chi^2{}_{v=12} \qquad \alpha = \frac{v}{2} \Rightarrow 6 = \frac{v}{2} \Rightarrow v = 2(6) = 12$

$$\chi^2{}_{1-\frac{\alpha}{2}, 12} < \frac{2}{\beta} \sum_{i=1}^{6} X_i < \chi^2{}_{\frac{\alpha}{2}, 12}$$

$$\frac{2\sum_{i=1}^{6} X_i}{\chi^2{}_{\frac{\alpha}{2}, 12}} < \beta < \frac{2\sum_{i=1}^{6} X_i}{\chi^2{}_{1-\frac{\alpha}{2}, 12}}$$

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