# Orthogonal Trajectories

Recall that two non-vertical lines $l_1 \ and \ l_2$ with slopes $m_1 \ and \ m_2$, respectively, are orthogonal if and only if $m_1 = \frac{-1}{m_2}$.

Let $F(x, y, c_1) = 0$ and $G(x, y, c_2) = 0$ be two families of curves in a plane such that their tangent lines are orthogonal at each point of intersection, then the two families are said to be orthogonal trajectories of each other.

If the D. E. of one family is $\frac{dy}{dx} = f(x, y)$,

then the D. E. of the second is $\frac{dy}{dx} = \frac{-1}{f(x,y)}$.

Example 1

Find the orthogonal trajectories for the family of circles

$$x^2 + (y-c)^2 = c^2. \qquad (1)$$

Solution. Let us rewrite equation (1) on the form

$$x^2 + y^2 - 2cy = 0. \qquad (2)$$

Differentiating equation (2) with respect to $x$ we get

$$2x + 2yy' - 2cy' = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{c-y}$$

From (2) we have $c = \frac{x^2+y^2}{2y}$, hence the differential equation of the given family is

$$\frac{dy}{dx} = \frac{2xy}{x^2-y^2} = f(x,y)$$

Therefore the differential equation of the orthogonal family is $\frac{dy}{dx} = \frac{-1}{f(x,y)} \Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$

$$or \quad (x^2 - y^2)dx + 2xydy = 0 \qquad (4)$$

which is homogeneous D.E.

To solve equation (4)

let $y = ux \Rightarrow dy = udx + xdu$ hence equation (4) becomes

$$(x^2 - x^2u^2)dx + 2x^2u(udx + xdu) = 0$$

$$\Rightarrow x^2(1+u^2)dx + 2x^3udu = 0$$

$$\Rightarrow x^2(1+u^2)dx = -2x^3udu$$

$$\Rightarrow -\frac{1}{x}dx = \frac{2u}{1+u^2}du \quad which\ is\ separable\ D.E.$$

Integrating both sides we obtain

$$\ln|x| + \ln(1+u^2) = c$$

$$or \quad \ln[x(1+\tfrac{y^2}{x^2})] = c \;\Rightarrow\; x^2 + y^2 = c_1 x,$$

where $c_1 = e^c$.

Example 2.

Find the member of the orthogonal trajectories of the family of curves $y^2 = c\,x^3$ which passes through the point A(2,0).

Solution. Since $y^2 = c\,x^3 \Rightarrow 2yy' = 3cx^2$.

But $c = \frac{y^2}{x^3} \Rightarrow \frac{dy}{dx} = \frac{3y}{2x} = f(x, y)$, this is the D. E. of the given family, hence the D. E. of the orthogonal family is

$$\frac{dy}{dx} = \frac{-1}{f(x,y)} \quad or \quad \frac{dy}{dx} = \frac{-2x}{3y} \quad which\ is\ separable\ D.\ E.$$

Integrating both sides we obtain

$$x^2 + \tfrac{3}{2} y^2 = c.$$

Since the curve passes through A(2,0), substituting this point in the last equation implies $c = 4$, hence the member is $x^2 + \tfrac{3}{2} y^2 = 4.$

Example 3

Find the orthogonal trajectories for the family of the hyperbolas $x^2 + 2xy - y^2 + 4x - 4y = c.$

Example 4

Find the orthogonal trajectories for the family of the curves

$$x^3 + 3xy^2 = c, \quad c \neq 0.$$