Chapter 2
Mathematical Expectation:
2.1 Mean of a Random Variable:

Definition 1:

Let X be a random variable with a probability distribution f(x). The mean (or expected value) of X is denoted by μ_X (or E(X)) and is defined by:

$$E(X) = \mu_X = \begin{cases} \sum_{all\ x} x\, f(x)\ ; & if\ \ X\ \ is\ discrete \\ \int_{-\infty}^{\infty} x\, f(x)\, dx\ ; & if\ \ X\ \ is\ continuous \end{cases}$$

Example 1: (Example 4 in ch1)

A shipment of 8 similar microcomputers to a retail outlet contains 3 that are defective and 5 are non-defective. If a school makes a random purchase of 2 of these computers, find the expected number of defective computers purchased

Solution:

Let X = the number of defective computers purchased.

In this example, we found that the probability distribution of X is:

x	0	1	2
f(x)=p(X=x)	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$

or:

$$f(x)=P(X=x)=\begin{cases}\dfrac{\binom{3}{x}\times\binom{5}{2-x}}{\binom{8}{2}}; & x=0,1,2\\ 0; & otherwise\end{cases}$$

The expected value of the number of defective computers purchased is the mean (or the expected value) of X, which is:

$$E(X) = \mu_X = \sum_{x=0}^{2} x\, f(x)$$

$$= (0)\, f(0) + (1)\, f(1) + (2)\, f(2)$$

$$= (0)\frac{10}{28} + (1)\frac{15}{28} + (2)\frac{3}{28}$$

$$= \frac{15}{28} + \frac{6}{28} = \frac{21}{28} = 0.75 \qquad \text{(computers)}$$

Example 2:

Let X be a continuous random variable that represents the life (in hours) of a certain electronic device. The pdf of X is given by:

$$f(x) = \begin{cases} \dfrac{20{,}000}{x^3} \ ; & x > 100 \\ 0 \ ; & elsewhere \end{cases}$$

Find the expected life of this type of devices.

Solution:

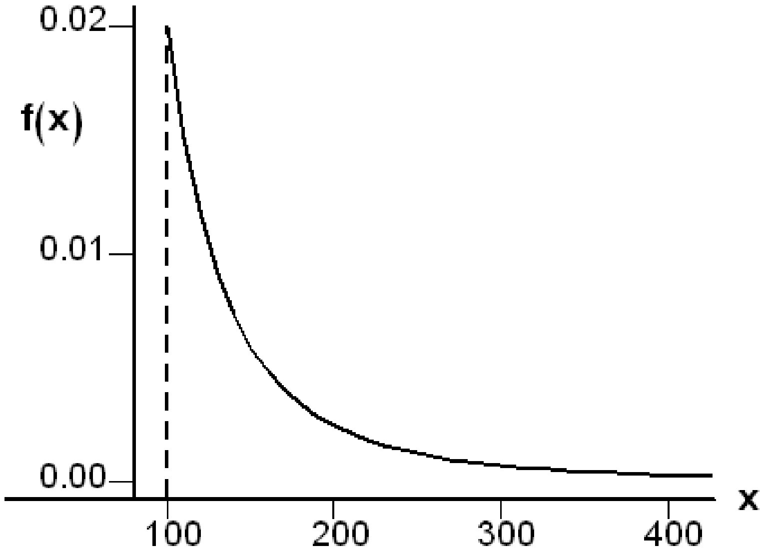
$$E(X) = \mu_X = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{100}^{\infty} x \frac{20000}{x^3}\, dx$$

$$= 20000 \int_{100}^{\infty} \frac{1}{x^2}\, dx$$

$$= 20000 \left[-\frac{1}{x} \Big|_{x=100}^{x=\infty} \right]$$

$$= -20000 \left[0 - \frac{1}{100} \right] = 200 \quad \text{(hours)}$$



Therefore, we expect that this type of electronic devices to last, on average, 200 hours.

Theorem 2.1:

Let X be a random variable with a probability distribution f(x), and let g(X) be a function of the random variable X. The mean (or expected value) of the random variable g(X) is denoted by $\mu_{g(X)}$ (or E[g(X)]) and is defined by:

$$E[g(X)] = \mu_{g(X)} = \begin{cases} \sum_{all\ x} g(x)\, f(x)\,; & if\ \ X\ is\ discrete \\ \int_{-\infty}^{\infty} g(x)\, f(x)\, dx\,; & if\ \ X\ is\ continuous \end{cases}$$

Example 3:

Let X be a discrete random variable with the following probability distribution

x	0	1	2
f(x)	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$

Find E[g(X)], where $g(X)=(X-1)^2$.

Solution:

$g(X)=(X-1)^2$

$$E[g(X)]=\mu_{g(X)}=\sum_{x=0}^{2} g(x)\,f(x)=\sum_{x=0}^{2}(x-1)^2\,f(x)$$

$$=(0-1)^2 f(0)+(1-1)^2 f(1)+(2-1)^2 f(2)$$

$$=(-1)^2\frac{10}{28}+(0)^2\frac{15}{28}+(1)^2\frac{3}{28}$$

$$=\frac{10}{28}+0+\frac{3}{28}=\frac{10}{28}$$

Example 4:

In Example 2, find $E\left(\frac{1}{X}\right)$.

Solution:

$$f(x) = \begin{cases} \frac{20,000}{x^3} \ ; \ x > 100 \\ 0 \ ; \ elsewhere \end{cases}$$

$$g(X) = \frac{1}{X}$$

$$E\left(\frac{1}{X}\right) = E[g(X)] = \mu_{g(X)} = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx = \int_{-\infty}^{\infty} \frac{1}{x} f(x)\, dx$$

$$= \int_{100}^{\infty} \frac{1}{x} \frac{20000}{x^3} dx = 20000 \int_{100}^{\infty} \frac{1}{x^4} dx = \frac{20000}{-3} \left[\frac{1}{x^3} \Bigg|_{x=100}^{x=\infty} \right]$$

$$= \frac{-20000}{3} \left[0 - \frac{1}{1000000} \right] = 0.0067$$

2.2 Variance (of a Random Variable):

The most important measure of variability of a random variable X is called the variance of X and is denoted by Var(X) or σ_X^2 .

Definition 2:

Let X be a random variable with a probability distribution f(x) and mean μ. The variance of X is defined by:

$$V(x) = \sigma^2 = E(x-\mu)^2 = \sum_{\forall x}(x-\mu)^2 f(x) = E(X^2) - (E(X))^2 \quad if\ x\ is\ discrete \ (2)$$

$$V(x) = \sigma^2 = E(x-\mu)^2 = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx = E(X^2) - (E(X))^2\ if\ x\ is\ continuous \ (3)$$

$$E(x^2) = \begin{cases} \sum x^2 f(x) & if\ x\ is\ discrete \\ \int_{-\infty}^{\infty} x^2 f(x)dx & if\ x\ is\ continuous \end{cases}$$

Definition 3:

The positive square root of the variance of X, $\sigma_X = \sqrt{\sigma_X^2}$,is called the **standard deviation** of X.

Note:

Var(X)=E[g(X)], where g(X)=$(X-\mu)^2$

Theorem 2.2:

The variance of the random variable X is given by:

$$\operatorname{Var}(X) = \sigma_X^2 = E(X^2) - \mu^2$$

where
$$E(X^2) = \begin{cases} \sum_{all\ x} x^2 f(x)\,; & if\ \ X\ \ is\ discrete \\ \int_{-\infty}^{\infty} x^2 f(x)\,dx\,; & if\ \ X\ \ is\ continuous \end{cases}$$

Example 5:

Let X be a discrete random variable with the following probability distribution

x	0	1	2	3
f(x)	0.15	0.38	0.10	0.01

Find Var(X)= σ_X^2 .

Solution:

$$\mu = \sum_{x=0}^{3} x\, f(x) = (0)\ f(0) + (1)\ f(1) + (2)\ f(2) + (3)\ f(3)$$

$$= (0)\ (0.51) + (1)\ (0.38) + (2)\ (0.10) + (3)\ (0.01) = 0.61$$

1. First method:

$$\mathrm{Var}(X) = \sigma_X^2 = \sum_{x=0}^{3} (x-\mu)^2 f(x)$$

$$= \sum_{x=0}^{3} (x-0.61)^2 f(x)$$

$$= (0-0.61)^2 f(0) + (1-0.61)^2 f(1) + (2-0.61)^2 f(2) + (3-0.61)^2 f(3)$$

$$= (-0.61)^2 (0.51) + (0.39)^2 (0.38) + (1.39)^2 (0.10) + (2.39)^2 (0.01)$$

$$= 0.4979$$

2. Second method:

$$\mathrm{Var}(X) = \sigma_X^2 = E(X^2) - \mu^2$$

$$E(X^2) = \sum_{x=0}^{3} x^2 f(x) = (0^2)\ f(0) + (1^2)\ f(1) + (2^2)\ f(2) + (3^2)\ f(3)$$

$$= (0)\ (0.51) + (1)\ (0.38) + (4)\ (0.10) + (9)\ (0.01) = 0.87$$

$$\mathrm{Var}(X) = \sigma_X^2 = E(X^2) - \mu^2 = 0.87 - (0.61)^2 = 0.4979$$

Example 6:

Let X be a continuous random variable with the following pdf:

$$f(x)=\begin{cases}2(x-1) \; ; \; 1<x<2 \\ 0 \; ; \; elsewhere\end{cases}$$

Find the mean and the variance of X.

Solution:

$$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{1}^{2} x[2(x-1)]\, dx = 2\int_{1}^{2} x(x-1)\, dx = 5/3$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)\, dx = \int_{1}^{2} x^2[2(x-1)]\, dx = 2\int_{1}^{2} x^2(x-1)\, dx = 17/6$$

$$Var(X) = \sigma_X^2 = E(X^2) - \mu^2 = 17/6 - (5/3)^2 = 1/8$$

2.3 Means and Variances of Linear Combinations of Random Variables:

If X_1, X_2, …, X_n are n random variables and a_1, a_2, …, a_n are constants, then the random variable :

$$Y = \sum_{i=1}^{n} a_i X_i = a_1 X_1 + a_2 X_2 + \cdots + a_n X_n$$

is called a linear combination of the random variables $X_1, X_2, \ldots, X_n$.

Theorem 2.3:

If X is a random variable with mean $\mu = E(X)$, and if a and b are constants, then:

$$E(aX \pm b) = a\,E(X) \pm b$$

$$\Leftrightarrow$$

$$\mu_{aX \pm b} = a\,\mu_X \pm b$$

Corollary 1: $E(b) = b$ (a=0 in Theorem 4.5)

Corollary 2: $E(aX) = a\,E(X)$ (b=0 in Theorem 4.5)

Example 7:

Let X be a random variable with the following probability density function:

$$f(x)=\begin{cases}\frac{1}{3}x^2 \; ; -1<x<2\\ 0 \; ; \; elsewhere\end{cases}$$

Find E(4X+3).

Solution:

$$\mu=E(X)=\int_{-\infty}^{\infty} x\, f(x)\,dx = \int_{-1}^{2} x[\frac{1}{3}x^2]\,dx = \frac{1}{3}\int_{-1}^{2} x^3\,dx = \frac{1}{3}\left[\frac{1}{4}x^4 \Big|_{x=-1}^{x=2}\right] = 5/4$$

E(4X+3) = 4 E(X)+3 = 4(5/4) + 3=8

Another solution:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f(x)\,dx \quad ; g(X) = 4X+3$$

$$E(4X+3) = \int_{-\infty}^{\infty} (4x+3)\, f(x)\,dx = \int_{-1}^{2} (4x+3)\,[\frac{1}{3}x^2]\,dx = \cdots = 8$$

Theorem 2.4:

If X_1, X_2, …, X_n are n random variables and a_1, a_2, …, a_n are constants, then:

$$E(a_1X_1+a_2X_2+ \dots +a_nX_n) = a_1E(X_1)+ a_2E(X_2)+ \dots+a_nE(X_n)$$

$$\Leftrightarrow$$

$$E(\sum_{i=1}^{n} a_i X_i) = \sum_{i=1}^{n} a_i E(X_i)$$

If X an Y are independent then for any functions h and g

$$\mathbf{E[h(X) . g(Y)]= E(h(X)) . E(g(Y))}$$

Corollary: **If X, and Y are random variables, then:**

$$\mathbf{E(X \pm Y) = E(X) \pm E(Y)}$$

Theorem 2.5:

If X is a random variable with variance $Var(X) = \sigma_X^2$ and if a and b are constants, then:

$$Var(aX \pm b) = a^2 Var(X)$$

$$\Leftrightarrow$$

$$\sigma_{aX+b}^2 = a^2 \sigma_X^2$$

Theorem 2.6:

If $X_1, X_2, \ldots, X_n$ are n <u>independent</u> random variables and $a_1, a_2, \ldots, a_n$ are constants, then:

$$Var(a_1X_1+a_2X_2+\ldots+a_nX_n) = a_1^2\, Var(X_1) + a_2^2\, Var(X_2) + \ldots + a_n^2\, Var(X_n)$$

$$\Leftrightarrow$$

$$Var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2\, Var(X_i)$$

$$\Leftrightarrow$$

$$\sigma^2_{a_1X_1+a_2X_2+\ldots+a_nX_n} = a_1^2\sigma^2_{X_1} + a_2^2\sigma^2_{X_2} + \ldots + a_n^2\sigma^2_{X_n}$$

Corollary:

If X, and Y are <u>independent</u> random variables, then:

- $Var(aX+bY) = a^2\, Var(X) + b^2\, Var(Y)$
- $Var(aX-bY) = a^2\, Var(X) + b^2\, Var(Y)$
- $Var(X \pm Y) = Var(X) + Var(Y)$

Example 8:

Let X, and Y be two independent random variables such that $E(X)=2$, $Var(X)=4$, $E(Y)=7$, and $Var(Y)=1$. Find:

1. $E(3X+7)$ and $Var(3X+7)$
2. $E(5X+2Y-2)$ and $Var(5X+2Y-2)$.

Solution:

1. $E(3X+7) = 3E(X)+7 = 3(2)+7 = 13$

   $Var(3X+7)= (3)^2 Var(X)=(3)^2 (4) =36$

2. $E(5X+2Y-2)= 5E(X) + 2E(Y) -2= (5)(2) + (2)(7) - 2= 22$

   $Var(5X+2Y-2)= Var(5X+2Y)= 5^2 Var(X) + 2^2 Var(Y)$

   $= (25)(4)+(4)(1) = 104$

Example 9:

The probability distribution for company **A** is given by:

X	1	2	3
f(x)	0.3	0.4	0.3

and for company **B** is given by:

Y	0	1	2	3	4
f(y)	0.2	0.1	0.3	0.3	0.1

Show that the variance of the probability distribution for company **B** is greater than that of company **A**.

Solution:

X	1	2	3	$\sum$
f(x)	0.3	0.4	0.3	1
x f(x)	0.3	0.8	0.9	2
f(x)x^2	0.3	1.6	2.7	4.6

$$\sigma^2 = E(x^2) - (E(x))^2 = 4.6 - 4 = 0.6, \sigma = .77$$

Y	0	1	2	3	4	$\sum$
f(y)	0.2	0.1	0.3	0.3	0.1	1
Y f(y)	0	0.1.	0.6	0.9	0.4	2
y^2 f(y)	0	0.1	1.2	2.7	1.6	5.6

$$\sigma^2 = E(y^2) - (E(y))^2 = 5.6 - 4 = 1.6, \sigma = 1.26$$

Note that σ_y^2 is greater than σ_x^2.

Problem 4 Let X have a mixed distribution $F(X)$ writen uniquely as

$$F(X) = cF_1(X) + (1-c)F_2(X)$$

where F_1 is the distribution function of a discrete random variable X_1 and F_2 is the distribution function of a continuous random variable X_2. Then $E(X^2)$ is

(a) $cE(X_1) + (1-c)E(X_2)$

(b) $E(X_1) + E(X_2)$

(c) $cE(X_1^2) + (1-c)E(X_2^2)$

(d) $E(X_1^2) + E(X_2^2)$

Solved Problems

4.15 The density function of the continuous random variable X, the total number of hours, ***in units of 100 hours***, that a family runs a vacuum cleaner over a period of one year, is given in Exercise 3.7 on page 88 as
Find the average number of hours per year that families run their vacuum cleaners.

Solution

4.15 $E(X) = \int_0^1 x^2\, dx + \int_1^2 x(2-x)\, dx = 1$. Therefore, the average number of hours per year is $(1)(100) = 100$ hours.

4.34 Let X be a random variable with the following probability distribution:
Find the standard deviation of *X*.

Solution

x	**-2**	**3**	**5**
f(x)	**0.3**	**0.2**	**0.5**

4.34 $\mu = (-2)(0.3) + (3)(0.2) + (5)(0.5) = 2.5$ and
$E(X^2) = (-2)^2(0.3) + (3)^2(0.2) + (5)^2(0.5) = 15.5$.
So, $\sigma^2 = E(X^2) - \mu^2 = 9.25$ and $\sigma = 3.041$.

4.36 Suppose that the probabilities are 0.4. 0.3, 0.2, and 0.1, respectively, that 0, 1, 2. or 3 power failures will strike a certain subdivision in any given year. Find the mean and variance of the random variable *X* representing the number of power failures striking this subdivision.

Solution

4.36 $\mu = (0)(0.4) + (1)(0.3) + (2)(0.2) + (3)(0.1) = 1.0$,
and $E(X^2) = (0)^2(0.4) + (1)^2(0.3) + (2)^2(0.2) + (3)^2(0.1) = 2.0$.
So, $\sigma^2 = 2.0 - 1.0^2 = 1.0$.

4.43 The length of time, in minutes, for an airplane to obtain clearance for take off at a certain airport is a random variable *Y* = 3X — 2, where X has the density function

$$f(x) = \begin{cases} \frac{1}{4} e^{-x/4}, & x > 0 \\ 0, & eleswhere \end{cases}$$

Find the mean and variance of the random variable *Y.*

Solution

4.43 $\mu_Y = E(3X - 2) = \frac{1}{4}\int_0^\infty (3x-2)e^{-x/4}\,dx = 10$. So
$\sigma_Y^2 = E\{[(3X-2) - 10]^2\} = \frac{9}{4}\int_0^\infty (x-4)^2 e^{-x/4}\,dx = 144$.

4.50 On a laboratory assignment, if the equipment is working, the density function of the observed outcome, X. is

$$\begin{cases} 2(1-x), & 0 < x < 1 \\ 0, & otherwise \end{cases}$$

Find the variance and standard deviation of *X*.

Solution

4.50 $E(X) = 2\int_0^1 x(1-x)\,dx = 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{3}$ and
$E(X^2) = 2\int_0^1 x^2(1-x)\,dx = 2\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{1}{6}$. Hence,
$Var(X) = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}$, and $\sigma = \sqrt{1/18} = 0.2357$.

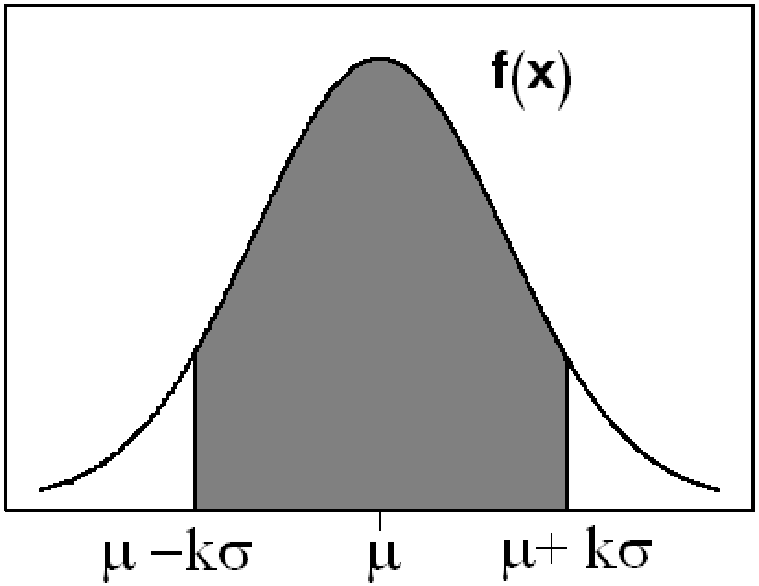
2.4 Chebyshev's Theorem:

* Suppose that X is any random variable with mean E(X)=μ and variance Var(X)= σ^2 and standard deviation σ.

* Chebyshev's Theorem gives a conservative estimate of the probability that the random variable X assumes a value within k standard deviations (kσ) of its mean μ, which is P(μ– kσ <X< μ +kσ).

* $P(\mu - k\sigma < X < \mu + k\sigma) \approx 1 - \frac{1}{k^2}$

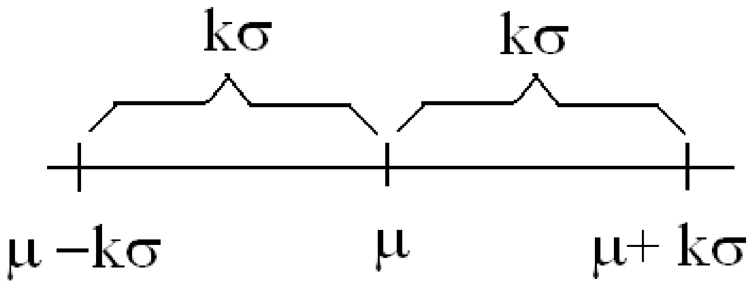
$$\text{area} = P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$



Theorem 2.7:(Chebyshev's Theorem)

Let X be a random variable with mean E(X)=μ and variance Var(X)=σ^2, then for k>1, we have:

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2} \Leftrightarrow \quad P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$



Example 10

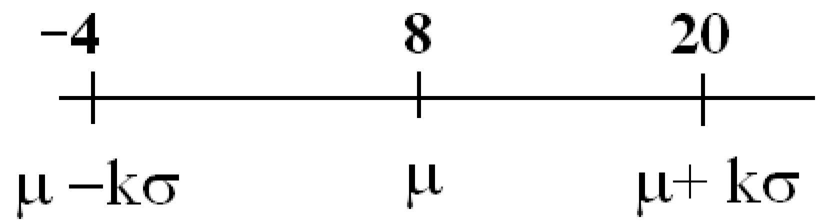
Let X be a random variable having an unknown distribution with mean μ=8 and variance σ^2=9 (standard deviation σ=3). Find the following probability:

(a) $P(-4 < X < 20)$

(b) $P(|X-8| \geq 6)$

Solution:

(a) $P(-4 < X < 20) = ??$



$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$

$(-4 < X < 20) = (\mu - k\sigma < X < \mu + k\sigma)$

$-4 = \mu - k\sigma \Leftrightarrow -4 = 8 - k(3)$ or $20 = \mu + k\sigma \Leftrightarrow 20 = 8 + k(3)$

$\Leftrightarrow -4 = 8 - 3k$ $\quad\quad \Leftrightarrow 20 = 8 + 3k$

$\Leftrightarrow 3k = 12$ $\quad\quad \Leftrightarrow 3k = 12$

$\Leftrightarrow k = 4$ $\quad\quad \Leftrightarrow k = 4$

$$1 - \frac{1}{k^2} = 1 - \frac{1}{16} = \frac{15}{16}$$

Therefore, $P(-4 < X < 20) \geq \frac{15}{16}$, and hence, $P(-4 < X < 20) \approx \frac{15}{16}$ (approximately)

(b) $P(|X-8| \geq 6) = ??$

$P(|X-8| \geq 6) = 1 - P(|X-8| < 6)$

$P(|X-8| < 6) = ??$

$P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$

$(|X-8| < 6) = (|X-\mu| < k\sigma)$

$6 = k\sigma \Leftrightarrow 6 = 3k \Leftrightarrow k = 2$

$$1 - \frac{1}{k^2} = 1 - \frac{1}{4} = \frac{3}{4}$$

$P(|X-8| < 6) \geq \frac{3}{4} \Leftrightarrow 1 - P(|X-8| < 6) \leq 1 - \frac{3}{4}$

$\Leftrightarrow 1 - P(|X-8| < 6) \leq \frac{1}{4}$

$\Leftrightarrow P(|X-8| \geq 6) \leq \frac{1}{4}$

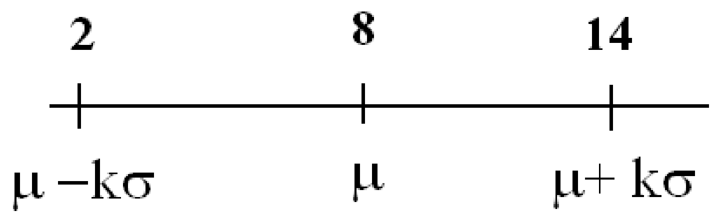
Therefore, $P(|X-8| \geq 6) \approx \frac{1}{4}$ (approximately)

Another solution for part (b):

$P(|X-8| < 6) = P(-6 < X-8 < 6)$

$= P(-6+8 < X < 6+8)$

$= P(2 < X < 14)$



$(2 < X < 14) = (\mu - k\sigma < X < \mu + k\sigma)$

$2 = \mu - k\sigma \Leftrightarrow 2 = 8 - k(3) \Leftrightarrow 2 = 8 - 3k \Leftrightarrow 3k = 6 \Leftrightarrow k = 2$

$$1 - \frac{1}{k^2} = 1 - \frac{1}{4} = \frac{3}{4}$$

$P(2 < X < 14) \geq \frac{3}{4} \Leftrightarrow P(|X-8| < 6) \geq \frac{3}{4}$

$\Leftrightarrow 1 - P(|X-8| < 6) \leq 1 - \frac{3}{4}$

$\Leftrightarrow 1 - P(|X-8| < 6) \leq \frac{1}{4}$

$\Leftrightarrow P(|X-8| \geq 6) \leq \frac{1}{4}$

Therefore, $P(|X-8| \geq 6) \approx \frac{1}{4}$ (approximately)