First order linear differential equation

A linear first order DE has the general form

$$a_1(x)\frac{dy}{dx} + a_2(x)y = g(x)$$

Or equivalently ,

$$\frac{dy}{dx} + p(x)y = f(x) \qquad (1)$$

We seek a solution of (1) defined on some interval I on which p and f are continuous.

It is easy to see that Equation (1) can be converted to an exact DE by using the integrating factor:

$$\mu(x) = e^{\int p(x)dx}$$

Multiplying both sides of Equation (1) by $\mu(x)$, we obtain

$$e^{\int p(x)dx}\frac{dy}{dx}+e^{\int p(x)dx}p(x)y=e^{\int p(x)dx}f(x)$$

Or

$$\frac{d}{dx}\left\{e^{\int p(x)dx}y\right\}=e^{\int p(x)dx}f(x)$$

That is $$\frac{d}{dx}\{\mu(x)\,y\}=\mu(x)\,f(x) \qquad (2)$$

Integrating both sides of (2) we get

$$\mu(x)\,y=c+\int \mu(x)\,f(x)\,dx$$

Or

$$y=\tfrac{1}{\mu(x)}\left[c+\int \mu(x)\,f(x)\,dx\right]$$

Example1: Solve the DE

$$x\frac{dy}{dx}-4y=x^6e^x,\ x>0.$$

Dividing both sides by x we get

$$\frac{dy}{dx}-\frac{4}{x}y=x^5e^x$$

$$\Rightarrow p(x)=-\frac{4}{x}\Rightarrow \mu(x)=e^{\int -\frac{4}{x}dx}=x^{-4}$$

Multiply both sides of (1) *by* x^{-4} *to get*

$$\Rightarrow x^{-4}\frac{dy}{dx}+(-4x^{-5})y=xe^x$$

$$Or\ \ \frac{d}{dx}\left(x^{-4}y\right)=xe^x$$

Which implies

$$x^{-4}y=\int xe^x dx$$

$$=xe^x-e^x+c$$

$$Or\ \ y=x^4\left(xe^x-e^x+c\right)$$

Example 2: Solve the DE:

$$(1-\cos x)dy+(2y\sin x-\tan x)dx=0$$

After rearranging, the equation becomes

$$\frac{dy}{dx}+\frac{2\sin x}{1-\cos x}y=\tan x \qquad (1).$$

$$\Rightarrow p(x)=\frac{2\sin x}{1-\cos x}\Rightarrow \mu(x)=e^{\int\frac{2\sin x}{1-\cos x}dx}=(1-\cos x)^2$$

Multiply both sides of (1) *by* $\mu(x)$ *to get*

$$\Rightarrow (1-\cos x)^2\frac{dy}{dx}+2\sin x(1-\cos x)y=\tan x(1-\cos x)^2$$

$$Or\ \ \frac{d}{dx}\left\{(1-\cos x)^2 y\right\}=\tan x(1-\cos x)^2$$

Which implies

$$(1-\cos x)^2 y=\int \tan x(1-\cos x)^2 dx$$

$$=-\ln|\cos x|+2\cos x+\tfrac{1}{2}\sin^2 x+c$$

Example 3

Solve the initial value problem

$$xy' - 2y = 5x^2, \quad y(1) = 2,$$

First put the equation in the standard form:

$$y' - \tfrac{2}{x} y = 5x, \quad \text{for } x \neq 0$$

Then

$$p(x) = \tfrac{-2}{x} \Rightarrow \mu(x) = e^{-\int \frac{2}{x} dx} = e^{-2\ln|x|} = e^{\ln\left(x^{-2}\right)} = \frac{1}{x^2}$$

hence

$$\tfrac{1}{x^2} y = \left[c + \int \frac{5}{x} dx \right]$$

$$\Rightarrow y = 5x^2 \ln|x| + c\,x^2$$

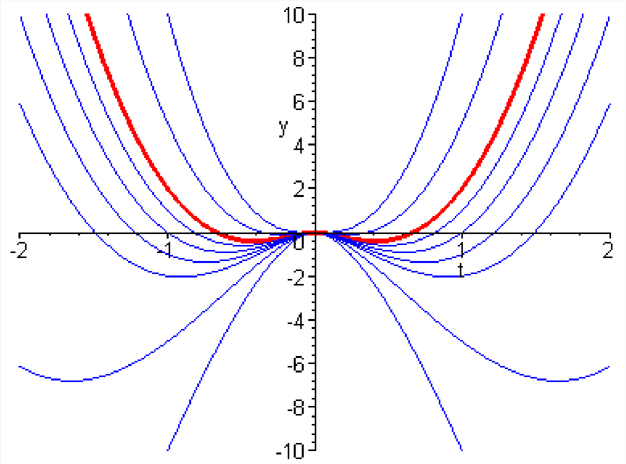
Using the initial condition $y(1) = 2$ in the general solution

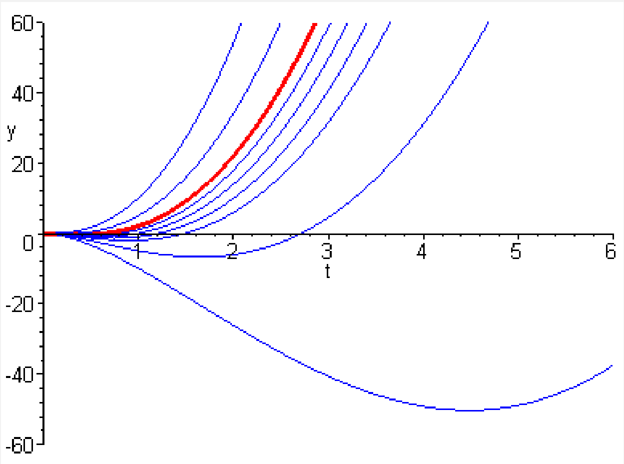
$$y = 5x^2 \ln|x| + cx^2,$$

it follows that

$$c = 2 \;\Rightarrow\; y = 5x^2 \ln|x| + 2x^2$$

The graphs below show several curves for different values of c, and a particular solution (in red) whose graph passes through the initial point (1,2).





Bernoulli's D. Equation

First order ODE of the form

$$\frac{dy}{dx} + p(x)y = f(x)\,y^n \qquad (1)$$

Where n is any real number different than 0 or 1 is called Bernoulli's DE, which can be reduced to a first order linear DE using by a suitable substitution.

Indeed, divide both sides of (1) by y^n to obtain

$$y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = f(x) \qquad (2)$$

$$Let\ \ u = y^{1-n} \Rightarrow \frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

$$\Rightarrow y^{-n}\frac{dy}{dx} = (1-n)\frac{du}{dx}$$

Therefore (2) becomes

$$(1-n)\frac{du}{dx} + p(x)u = f(x)$$

Which is linear DE in u

Example1

Solve the DE

$$x\frac{dy}{dx} + y(1 - x^2 y) = 0 \qquad (1)$$

Rewrite Equation (1) in the standard form

$$x\frac{dy}{dx} + y = x^2 y^2 \qquad (2)$$

Now, (2) is a Bernoulli's equation. Dividing both sides of (2) by y^2 we get

$$xy^{-2}\frac{dy}{dx} + y^{-1} = x^2 \qquad (3)$$

$$Let\ \ u = y^{-1} \Rightarrow \frac{du}{dx} = -y^{-2}\frac{dy}{dx}$$

$$\Rightarrow y^{-2}\frac{dy}{dx} = -\frac{du}{dx}$$

Using these values in (3) we obtain

$$-x\frac{du}{dx} + u = x^2 \qquad (4)$$

Dividing (4) by $-x$ we obtain

$$\frac{du}{dx}+\left(\frac{-1}{x}\right)u=-x \qquad (5)$$

which is LDE.

$$\Rightarrow p(x)=\left(\frac{-1}{x}\right)\Rightarrow \mu(x)=e^{\int\frac{-1}{x}dx}=\frac{1}{x}$$

Multiplying both sides of (5) by $\mu(x)=\frac{1}{x}$ we obtain

$$\frac{1}{x}\frac{du}{dx}+\left(\frac{-1}{x^2}\right)u=-1$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{x}u\right)=-1\Rightarrow \frac{1}{x}u=c-x$$

$$\Rightarrow u=cx-x^2 \Rightarrow y^{-1}=cx-x^2$$

Or

$$y(cx-x^2)=1$$

Example 2

Solve the DE

$$y' + xy - xe^{-x^2} y^{-3} = 0 \qquad (1)$$

Rewrite Equation (1) in the standard form

$$y' + xy = xe^{-x^2} y^{-3} \qquad (2)$$

which is a Bernoulli's DE. Multiplying both sides of (2) by y^3 we get

$$y^3 y' + xy^4 = xe^{-x^2} \qquad (3)$$

$$Let \quad u = y^4 \Rightarrow \frac{du}{dx} = 4y^3 \frac{dy}{dx}$$

$$\Rightarrow y^3 \frac{dy}{dx} = \frac{1}{4} \frac{du}{dx}$$

Using these values in (3) we obtain

$$\frac{1}{4} \frac{du}{dx} + xu = xe^{-x^2} \qquad (4)$$

Multiplying (4) by 4 we obtain

$$\frac{du}{dx}+4xu=4xe^{-x^2} \qquad (5)$$

which is LDE.

$$\Rightarrow p(x)=4x \Rightarrow \mu(x)=e^{\int 4x dx}=e^{2x^2}$$

Multiplying both sides of (5) by $\mu(x)$ we obtain

$$e^{2x^2}\frac{du}{dx}+4xe^{2x^2}u=4xe^{x^2}$$

$$\Rightarrow \frac{d}{dx}(e^{2x^2}u)=4xe^{2x^2} \Rightarrow e^{2x^2}u=c+\int 4xe^{2x^2}dx$$

$$\Rightarrow u=e^{-2x^2}(c+e^{2x^2})$$

$$Or \quad y^4=ce^{-2x^2}+1.$$

Homework

Solve the DEs

$$(1)\quad \frac{x}{y}\frac{dy}{dx} + xy = 1$$

$$(2)\quad 3(1+x^3)y' = 2xy(y^3-1)$$