Definition

Let f be a function in two variables x and y, then the differential of f (df) is given by

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dx$$

Exact Equations

A first order ODE of the form

$$M(x,y)dx + N(x,y)dy = 0$$

is said to be **exact DE** if there is a function $f(x,y)$ satisfies:

$$df = M(x,y)dx + N(x,y)dy$$

That is $f_x(x,y) = M(x,y), \;\; f_y(x,y) = N(x,y)$

hence $f(x,y) = \int M(x,y)\,dx$

or $f(x,y) = \int N(x,y)\,dy$

then the solution of the DE is given implicitly by $f(x,y) = c$

Theorem

Suppose M, N, $\dfrac{\partial M}{\partial y}$ and $\dfrac{\partial N}{\partial x}$ are continuous on an open region R in the xy-plane. Then, the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

is an **exact** if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial X}, \quad \forall (x,y) \in R$$

Example 1

The DE $(x^2+5y)dx+(y^3+5x)dy=0$

is exact, since $M(x,y)=x^2+5y,\ N(x,y)=y^3+5x$

and $\dfrac{\partial M}{\partial y}=5=\dfrac{\partial N}{\partial x}$

While, the DE $(x^2+y^2)dx+(3y+x)dy=0$

is not exact, because

$$M(x,y)=x^2+y^2, N(x,y)=3y+x$$

and $\dfrac{\partial M}{\partial y}=2y\neq 1=\dfrac{\partial N}{\partial x}$

Example 2

Solve the differential equation

$$(y\cos x + 2xe^{y})dx + (\sin x + x^{2}e^{y} - 1)dy = 0$$

Here

$$M(x,y) = y\cos x + 2xe^{y},\ N(x,y) = \sin x + x^{2}e^{y} - 1$$

hence

$$M_{y}(x,y) = \cos x + 2xe^{y} = N_{x}(x,y) \Rightarrow \text{ODE is exact}$$

Thus $f(x,y) = \int f_{x}(x,y)dx = \int \left(y\cos x + 2xe^{y}\right)dx$

$$= y\sin x + x^{2}e^{y} + g(y)$$

But

$$\frac{\partial f}{\partial y} = N \Rightarrow \frac{\partial}{\partial y}\left(y\sin x + x^2e^y + g(y)\right) = \sin x + x^2e^y - 1$$

$$\Rightarrow \sin x + x^2y + g'(y) = \sin x + x^2e^y - 1$$

$$\Rightarrow g'(y) = -1$$

$$\Rightarrow g(y) = -y + c_1$$

Therefore, $f(x, y) = y\sin x + x^2e^y - y + c_1$

hence the solution is given implicitly by

$$y\sin x + x^2e^y - y + c_1 = c$$

or

$$y\sin x + x^2e^y - y = k$$

Example 3

Solve the following differential equation.

$$\frac{dy}{dx}+\frac{x+4y}{4x-y}=0 \iff (x+4y)dx+(4x-y)dy=0$$

Here we have

$$M(x,y)=x+4y,\ N(x,y)=4x-y$$

hence $\quad M_y(x,y)=4=N_x(x,y) \Rightarrow$ *the* DE is exact

(Also, it is homogeneous DE)

Thus, the solution is given by $f(x,y)=c$ where

$$f_x(x,y)=x+4y,\quad f_y(x,y)=4x-y$$

Hence $\quad f(x,y)=\int f_x(x,y)dx=\int\left(x+4y\right)dx=\frac{1}{2}x^2+4xy+g(y)$

But

$$\frac{\partial f}{\partial y} = N \Rightarrow \frac{\partial}{\partial y}\left(\frac{1}{2}x^2 + 4xy + g(y)\right) = 4x - y$$

$$\Rightarrow 4x + g'(y) = 4x - y$$

$$\Rightarrow g'(y) = -y$$

$$\Rightarrow g(y) = -\tfrac{1}{2}y^2 + c_1$$

Hence

$$f(x,y) = \frac{1}{2}x^2 + 4xy - \frac{1}{2}y^2 + c_1$$

It follows that the solution is given by

$$\frac{1}{2}x^2 + 4xy - \frac{1}{2}y^2 + c_1 = c$$

or

$$\frac{1}{2}x^2 + 4xy - \frac{1}{2}y^2 = k$$

Example 4

Solve the IVP

$$(1+\ln x+\frac{y}{x})dx=(1-\ln x)dy,\ \ y(1)=2$$

First, put the DE in the form

$$(1+\ln x+\frac{y}{x})dx+(\ln x-1)dy=0$$

Hence $M=(1+\ln x+\frac{y}{x}),\ N=(\ln x-1)$

$$\Rightarrow M_y=\frac{1}{x}=N_x\Rightarrow the\,DE\,is\,Exact$$

$$\Rightarrow\exists a\ function\,f(x,y)\ such\ that$$

$$f_x=M\ and\ \ f_y=N$$

Therefore $f(x,y)=\int N\,dy=\int(\ln x-1)\,dy$

$$= y\ln x - y + h(x)$$

$$But\ \ f_x = M \Rightarrow \frac{y}{x} + h'(x) = 1 + \ln x + \frac{y}{x}$$

$$\Rightarrow h'(x) = 1 + \ln x$$

$$\Rightarrow h(x) = \int (1+\ln x)dx$$

$$= x\ln x + c_1$$

$$\Rightarrow\ f(x,y) = y\ln x - y + x\ln x + c_1$$

thus, the solution is given by

$$y\ln x - y + x\ln x = k$$

$$Since\ \ y(1) = 2 \Rightarrow k = -2$$

$$\Rightarrow the\ solution\ is: y\ln x - y + x\ln x + 2 = 0$$

Integrating Factors

Sometimes, it is possible to convert a non-exact DE into an exact DE equation by multiplying it by a suitable function $\mu(x,y)$ (called an integrating factor) :

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Case 1: If $\frac{1}{N}(M_y - N_x) = f(x)$, that is it does not depend on y.

Then $\mu(x) = e^{\int f(x)\,dx}$.

Case 2: If $\frac{1}{M}(M_y - N_x) = g(y)$, that is it does not depend on x.

Then $\mu(y) = e^{-\int g(y)\,dx}$.

Example 5

The following DE is not exact

$$(3xy + y^2)dx + (x^2 + xy)dy = 0$$

Here, $M = 3xy + y^2 + 1,\ N = x^2 + xy$

$$\Rightarrow M_y = 3x + 2y,\ N_x = 2x + y$$

$$\Rightarrow \frac{1}{N}\left(M_y - N_x\right) = \frac{x+y}{x^2+xy} = \frac{1}{x},\ \textit{function of } x \textit{ alone}$$

$$\Rightarrow I.F.is\ \ \mu(x) = e^{\int \frac{1}{x}dx} = x$$

Multiplying the DE by $\mu(x) = x$ it becomes

$$(3x^2y + xy^2)dx + (x^3 + x^2y)dy = 0$$

Which is an exact DE.

Example 6

The following DE is not exact

$$6xydx+(4y+9x^2)dy=0$$

Here, $M=6xy,\ N=4y+9x^2$

$$\Rightarrow M_y=6x,\ N_x=18x$$

$$\Rightarrow \frac{1}{M}\left(M_y-N_x\right)=\frac{-12x}{6xy}=\frac{-2}{y},\ \textit{function of } y \textit{ alone}$$

$$\Rightarrow I.F.is\ \mu(y)=e^{\int\frac{2}{y}dy}=y^2$$

Multiplying the DE by $\mu(y)=y$ it becomes

$$6xy^3dx+(4y^3+9x^2y^2)dy=0$$

Which is an exact DE.