

Separable Differential Equations

A first order ordinary differential equation

$$\frac{dy}{dx} = F(x, y)$$

is said to be **Separable** if it can be written in the form

$$\frac{dy}{dx} = f(x)g(y)$$

Or equivalently: $p(x)dx = q(y)dy$.

In this case the solution is obtained by integrating both sides:

$$\int p(x) dx = \int q(y) dy$$

Example 1

Solve the following differential equation:

$$\frac{dy}{dx} = \frac{x^2 + 1}{y^2 - 1}$$

Separating variables, and integrating both sides we get

$$(y^2 - 1)dy = (x^2 + 1)dx$$

$$\int (y^2 - 1)dy = \int (x^2 + 1)dx$$

$$\frac{1}{3}y^3 - y = \frac{1}{3}x^3 + x + c$$

$$y^3 - 3y = x^3 + 3x + c.$$

Example 2:

Solve the differential equation

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}$$

Separating variables, then integrating both sides we obtain:

$$2(y-1)dy = (3x^2 + 4x + 2)dx$$

$$2\int (y-1)dy = \int (3x^2 + 4x + 2)dx$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$

Example 3

Solve the initial value problem:

$$y' = \frac{y \cos x}{1 + 3y^3}, \quad y(0) = 1$$

Separating variables and **integrating both sides**, we obtain

$$\frac{1 + 3y^3}{y} dy = \cos x dx$$

$$\int \left(\frac{1}{y} + 3y^2 \right) dy = \int \cos x dx$$

$$\ln|y| + y^3 = \sin x + c$$

Using the initial condition we get $c = 1$, hence

$$\ln y + y^3 = \sin x + 1$$

Example 4:

Solve the differential equation

$$\frac{dy}{dx} = \frac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$$

First, let us factorize the numerator and denominator by grouping to obtain

$$\frac{dy}{dx} = \frac{x(y + 3) - (y + 3)}{x(y - 2) + 4(y + 8)} = \frac{(y + 3)[x - 1]}{(y - 2)[x + 4]}$$

Now, separating variables, then integrating both sides we get:

$$\frac{y-2}{y+3} dy = \frac{x-1}{x+4} dx$$

$$\Rightarrow \int \frac{y-2}{y+3} dy = \int \frac{x-1}{x+4} dx$$

$$\Rightarrow \int \left[1 - \frac{5}{y+3} \right] dy = \int \left[1 - \frac{5}{x+4} \right] dx$$

$$\Rightarrow y - 5 \ln |y+3| = x - 5 \ln |x+4| + c.$$

Homework

Solve the DE

$$\frac{dy}{dx} = \sin x \left(\cos(2y) - \cos^2 y \right).$$

Separation by substitution

First order ordinary differential equations of the form

$$\frac{dy}{dx} = f(ax + by + c), \quad a, b, c \in \mathbb{R}$$

can be reduced to a **separable** DE by the substitution:

$$u = ax + by + c \Rightarrow \frac{du}{dx} = a + b \frac{dy}{dx}$$

Example 1

Solve $\frac{dy}{dx} = \tan^2(x + y)$.

Solution: Let $u = x + y \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 1$

Thus we have $\frac{du}{dx} = 1 + \tan^2(u) \Rightarrow \frac{du}{dx} = \sec^2(u)$

Which is a separable differential equation.

Separating variables and integrating both sides we obtain:

$$\cos^2(u) du = dx$$

$$\Rightarrow \int \cos^2(u) du = \int dx$$

$$\Rightarrow \int \frac{1}{2} (1 + \cos(2u)) du = \int dx$$

$$\Rightarrow \frac{1}{2} (u + \frac{1}{2} \sin(2u)) = x + c$$

$$\Rightarrow \frac{1}{2} (x + y + \frac{1}{2} \sin(2(x + y))) = x + c.$$

Example 2

Solve the DE

$$\frac{dy}{dx} - 2 = \sqrt{y - 2x + 3}.$$

Solution: Let $u = y - 2x + 3 \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + 2$

Separating variables and integrating both sides we obtain:

$$\frac{du}{dx} = \sqrt{u}$$

$$\Rightarrow u^{-\frac{1}{2}} du = dx$$

$$\Rightarrow 2u^{\frac{1}{2}} = x + c$$

$$\Rightarrow 2\sqrt{y - 2x + 3} = x + c.$$

Example 3

Solve the DE

$$\frac{dy}{dx} = \frac{1-x-y}{x+y} \iff \frac{dy}{dx} = \frac{1-(x+y)}{x+y}.$$

Solution: Let $u = x + y \implies \frac{dy}{dx} = \frac{du}{dx} - 1$

Using these values in the DE we obtain:

$$\frac{du}{dx} - 1 = \frac{1-u}{u} \implies \frac{du}{dx} = 1 + \frac{1-u}{u}$$

$$\implies \frac{du}{dx} = \frac{1}{u} \implies u du = dx$$

$$\implies \frac{1}{2} u^2 = x + c$$

$$\implies \frac{1}{2} (x + y)^2 = x + c.$$