Fourier Integral

The Fourier integral of a function f defined on $(-\infty, \infty)$ is given by

$$f(x) \approx \frac{1}{\pi}\int_0^\infty \{A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x)\}d\alpha,$$

where

$$A(\alpha) = \int_{-\infty}^{\infty} f(x)\cos(\alpha x)dx,$$

$$B(\alpha) = \int_{-\infty}^{\infty} f(x)\sin(\alpha x)dx.$$

Theorem. Let f and f' be piecewise continuous on every finite subinterval and f is absolutely integrable on the interval $(-\infty, \infty)$.

Then the Fourier integral of f converges to $f(x_0)$, if f is continuous at x_0 and converges to $\frac{f(x_0^+)+f(x_0^-)}{2}$ if f is discontinuous at x_0, where $f(x_0^+)$, $f(x_0^-)$ are the right and left hand limits of f at x_0.

Example 1. Let

$$f(x)=\begin{cases}0, & x<0,\\ 1, & 0<x<2,\\ 0, & x>2.\end{cases}$$

Find the Fourier integral of f, then deduce that $\int_0^\infty \frac{\sin\alpha}{\alpha}\,d\alpha=\frac{\pi}{2}$.

Solution.

$$A(\alpha)=\int_{-\infty}^{\infty} f(x)\cos(\alpha x)dx=\int_0^2\cos(\alpha x)dx=\frac{\sin(\alpha x)}{\alpha}\bigg]_0^2=\frac{\sin(2\alpha)}{\alpha}.$$

$$B(\alpha) = \int_{-\infty}^{\infty} f(x)\sin(\alpha x)dx = \int_{0}^{2}\sin(\alpha x)dx = -\frac{\cos(\alpha x)}{\alpha}\Big]_0^2 = \left(\frac{1-\cos(2\alpha)}{\alpha}\right).$$

Hence, the Fourier integral of f is

$$f(x) \approx \frac{1}{\pi}\int_0^{\infty}\left\{\frac{\sin(2\alpha)}{\alpha}\cos(\alpha x) + \left(\frac{1-\cos(2\alpha)}{\alpha}\right)\sin(\alpha x)\right\}d\alpha.$$

Now, the function f is continuous at $x=1$, therefore $x=1$ the integral on the R.H.S. converges to $f(1)=1$, thus

$$1 = \frac{1}{\pi}\int_0^{\infty}\left\{\frac{\sin(2\alpha)}{\alpha}\cos(\alpha) + \left(\frac{1-\cos(2\alpha)}{\alpha}\right)\sin(\alpha)\right\}d\alpha$$

$$= \frac{1}{\pi}\int_0^{\infty}\left\{\frac{2\sin(\alpha)\cos^2(\alpha)}{\alpha} + \left(\frac{1-\cos(2\alpha)}{\alpha}\right)\sin(\alpha)\right\}d\alpha$$

$$= \frac{1}{\pi}\int_0^{\infty}\frac{\sin\alpha}{\alpha}\left[2\cos^2(\alpha) + 1 - \cos(2\alpha)\right]d\alpha, \qquad \left(\ 2\cos^2(\alpha) = 1 + \cos(2\alpha)\right).$$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{\sin\alpha}{\alpha}d\alpha,$$

$$\Rightarrow \int_0^{\infty}\frac{\sin\alpha}{\alpha}d\alpha = \frac{\pi}{2}.$$

Example 2. Find the Fourier integral representation of

$$f(x)=\begin{cases}0, & x<-1,\\ x, & -1<x<\pi,\\ 0, & x>\pi.\end{cases}$$

Solution.

$$A(\alpha)=\int_{-\infty}^{\infty} f(x)\cos(\alpha x)dx=\int_{-1}^{\pi} x\cos(\alpha x)dx = x\frac{\sin(\alpha x)}{\alpha}\Big]_{-1}^{\pi}-\int_{-1}^{\pi}\frac{\sin(\alpha x)}{\alpha}dx$$

$$=\frac{\pi\sin(\alpha\pi)-\sin\alpha}{\alpha}+\frac{\cos(\alpha\pi)-\cos\alpha}{\alpha^2}.$$

$$B(\alpha)=\int_{-\infty}^{\infty} f(x)\sin(\alpha x)dx=\int_{-1}^{\pi} x\sin(\alpha x)dx=-\frac{x\cos(\alpha x)}{\alpha}\Big]_{-1}^{\pi}+\int_{-1}^{\pi}\frac{\cos(\alpha x)}{\alpha}dx$$

$$=-\frac{\pi\cos(\pi\alpha)}{\alpha}-\frac{\cos(\alpha)}{\alpha}+\frac{\sin(\pi\alpha)+\sin(\alpha)}{\alpha^2}.$$

Hence the Fourier integral of f is

$$f(x)\approx\frac{1}{\pi}\int_0^{\infty}\left\{\left(\frac{\pi\sin(\alpha\pi)-\sin\alpha}{\alpha}+\frac{\cos(\alpha\pi)-\cos\alpha}{\alpha^2}\right)\cos(\alpha x)+\left(-\frac{\pi\cos(\pi\alpha)}{\alpha}-\frac{\cos(\alpha)}{\alpha}+\frac{\sin(\pi\alpha)+\sin(\alpha)}{\alpha^2}\right)\sin(\alpha x)\right\}d\alpha.$$

If f is an even function, then the Fourier integral of f on $(-\infty, \infty)$ is the cosine integral

$$f(x) \approx \frac{1}{\pi}\int_0^{\infty} A(\alpha)\cos(\alpha x)\, d\alpha,$$

where

$$A(\alpha) = 2\int_0^{\infty} f(x)\cos(\alpha x)dx.$$

Similarly, if f is an odd function, then the Fourier integral of f on $(-\infty, \infty)$ is the sine integral

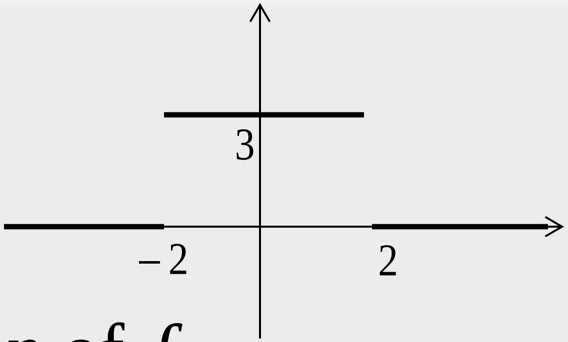
$$f(x) \approx \frac{1}{\pi}\int_0^{\infty} B(\alpha)\sin(\alpha x)\, d\alpha,$$

where

$$B(\alpha) = \int_{-\infty}^{\infty} f(x)\sin(\alpha x)dx.$$

Example 1. Let

$$f(x) = \begin{cases} 3, & |x| < 2, \\ 0, & |x| > 2. \end{cases}$$



Find the Fourier integral representation of f .

Solution.

From it's graph it follows that f is even. Hence

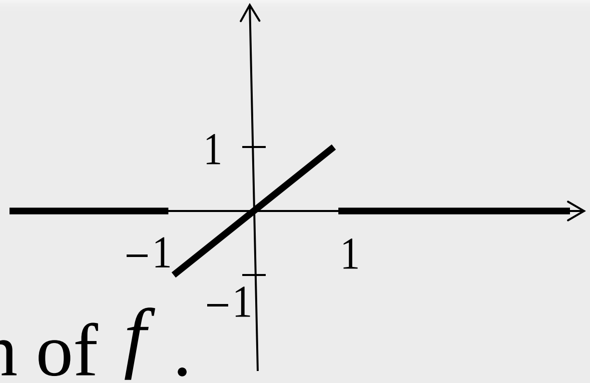
$$A(\alpha) = 2\int_0^\infty f(x)\cos(\alpha x)dx = 2\int_0^2 3\cos(\alpha x)dx = \frac{6\sin(\alpha x)}{\alpha}\Big]_0^2 = \frac{6\sin(2\alpha)}{\alpha},$$

thus, the Fourier integral representation of f is

$$f(x) \approx \frac{1}{\pi}\int_0^\infty A(\alpha)\cos(\alpha x)\, d\alpha = \frac{6}{\pi}\int_0^\infty \frac{\sin(2\alpha)}{\alpha}\cos(\alpha x)\, d\alpha.$$

Example 2. Let

$$f(x)=\begin{cases} x, & |x|<1, \\ 0, & |x|>1. \end{cases}$$



Find the Fourier integral representation of f.

Solution.

From it's graph it follows that f is odd. Hence

$$B(\alpha)=2\int_0^\infty f(x)\sin(\alpha x)dx$$

$$=2\int_0^1 x\sin(\alpha x)dx=\left.\frac{-2x\cos(\alpha x)}{\alpha}\right]_0^1+\int_0^1\frac{2\cos(\alpha x)}{\alpha}dx=\frac{-2\cos(\alpha)}{\alpha}+\frac{2\sin(\alpha)}{\alpha^2},$$

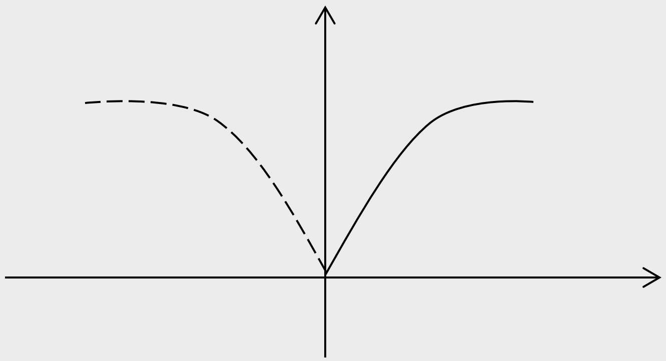
and the Fourier integral representation of f is

$$f(x)\approx\frac{1}{\pi}\int_0^\infty B(\alpha)\sin(\alpha x)\,d\alpha=\frac{1}{\pi}\int_0^\infty\left\{\frac{-2\cos(\alpha)}{\alpha}+\frac{2\sin(\alpha)}{\alpha^2}\right\}\sin(\alpha x)\,d\alpha.$$

Let f be a function defined only for $x > 0$, then it can be represented by a Fourier integral in two ways:

(1) As a cosine integral:

By defining f on $x < 0$ by $f(x) = f(-x)$, then this extension is an even function, hence we get



$$f(x) \approx \frac{1}{\pi}\int_0^\infty A(\alpha)\cos(\alpha x)\,d\alpha,$$

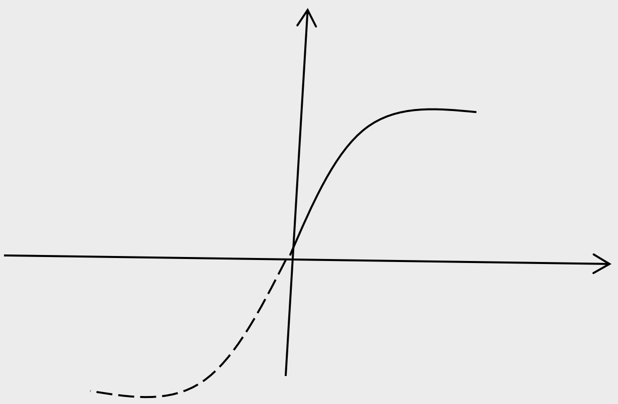
where

$$A(\alpha) = 2\int_0^\infty f(x)\cos(\alpha x)dx.$$

(2) As a sine integral:

By defining f on $x < 0$ by $f(x) = -f(-x)$, then this extension is an odd function, hence we get

$$f(x) \approx \frac{1}{\pi}\int_0^\infty B(\alpha)\sin(\alpha x)\,d\alpha,$$



where

$$B(\alpha) = 2\int_0^\infty f(x)\sin(\alpha x)dx.$$

Example. Let $f(x) = \begin{cases} x, & 0 < x < 1, \\ 0, & x > 1. \end{cases}$

Represent f (1) by a cosine integral and (2) by a sine integral.

Solution.

(1) $A(\alpha) = 2\int_0^\infty f(x)\cos(\alpha x)dx$

$$= 2\int_0^1 x\cos(\alpha x)dx = \frac{2x\sin(\alpha x)}{\alpha}\Bigg]_0^1 - \int_0^1 \frac{2\sin(\alpha x)}{\alpha}dx = \frac{2\sin(\alpha)}{\alpha} + \frac{2\{\cos(\alpha)-1\}}{\alpha^2},$$

hence, the cosine integral representation of f is

$$f(x) \approx \frac{1}{\pi}\int_0^\infty A(\alpha)\cos(\alpha x)\, d\alpha = \frac{1}{\pi}\int_0^\infty \left[\frac{2\sin(\alpha)}{\alpha} + \frac{2\{\cos(\alpha)-1\}}{\alpha^2}\right]\cos(\alpha x)\, d\alpha.$$

(2) $B(\alpha) = 2\int_0^\infty f(x)\sin(\alpha x)dx$

$$= 2\int_0^1 x\sin(\alpha x)dx = \frac{-2x\cos(\alpha x)}{\alpha}\Bigg]_0^1 + \int_0^1 \frac{2\cos(\alpha x)}{\alpha}dx = \frac{-2\cos(\alpha)}{\alpha} + \frac{2\sin(\alpha)}{\alpha^2},$$

hence, the sine integral representation of f is

$$f(x) \approx \frac{1}{\pi}\int_0^\infty B(\alpha)\sin(\alpha x)\, d\alpha = \frac{1}{\pi}\int_0^\infty \left\{\frac{-2\cos(\alpha)}{\alpha} + \frac{2\sin(\alpha)}{\alpha^2}\right\}\sin(\alpha x)\, d\alpha.$$