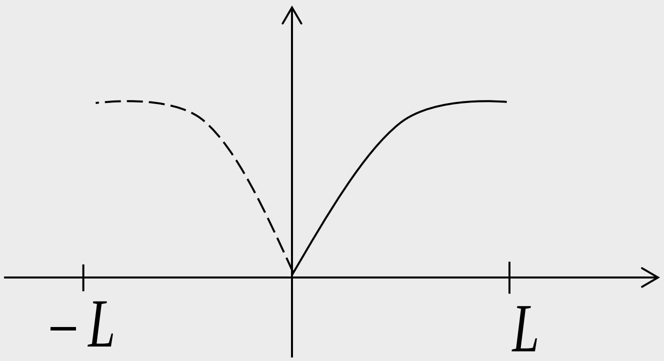
Half-range expansions

Let f be a function defined only for $0 < x < L.$

Then, f can be expanded in a trigonometric series in several ways:



(1) In a cosine series:

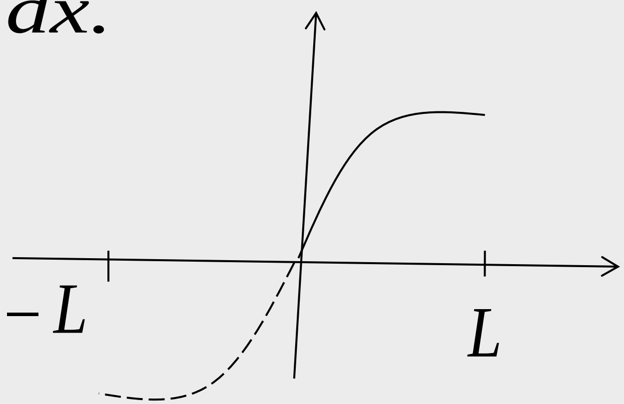
Define f on $-L < x < 0$ by $f(x) = f(-x)$, then the new function is even on $-L < x < L$, hence it can be expanded in the cosine series

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L} \right) \right\},$$

where

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx,$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx.$$



(2) In a sine series:

Define f on $(-L, 0)$

by $f(x) = -f(-x)$, then the new function is odd on $(-L, L)$, hence it can be expanded in the sine series

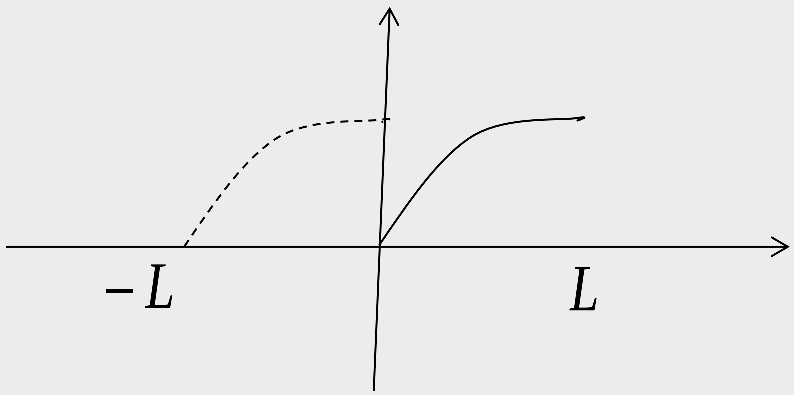
$$f(x) \approx \sum_{n=1}^{\infty}\left\{b_n \cos\left(\frac{n\pi x}{L}\right)\right\},$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

(3) In a Fourier series:

Define f on $(-L,0)$ by $f(x)=f(x+L)$, that is by considering f as a periodic function with period L. Hence f can be represented by the Fourier series



$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n \cos\left(\frac{2n\pi x}{L}\right) + b_n \sin\left(\frac{2n\pi x}{L}\right)\right\},$$

where

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx,$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2n\pi x}{L}\right)dx,$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2n\pi x}{L}\right)dx.$$

Example. Let $f(x) = x, \quad 0 < x < 1.$

Expand f in

(i) a cosine series

(ii) a sine series

(iii) a Fourier series.

Solution.

(1) By extending f to an even function on $(-1,1)$, with $L=1$, then we have

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx = 2\int_0^1 x\ dx = 1,$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = 2\int_0^1 x\ \cos(n\pi x)\,dx = 2x\frac{\sin n\pi x}{n\pi}\Bigg]_0^1 - \int_0^1 2\frac{\sin n\pi x}{n\pi}dx$$

$$= 2x\frac{\sin n\pi x}{n\pi}\Bigg]_0^1 + 2\frac{\cos n\pi x}{(n\pi)^2}\Bigg]_0^1 = 2\frac{\cos n\pi - 1}{(n\pi)^2} = \frac{2\{(-1)^n - 1\}}{(n\pi)^2}.$$

Therefore the Fourier series of f is

$$f(x) \approx \sum_{n=1}^{\infty}\left\{a_n \cos\left(\frac{n\pi x}{L}\right)\right\} = \sum_{n=1}^{\infty}\left\{\frac{2\{(-1)^n - 1\}}{(n\pi)^2}\cos(n\pi x)\right\}.$$

(2) By extending f to an odd function on $(-1,1)$, with $L=1$, we get

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = 2\int_0^1 x\,\sin(n\pi x)\,dx$$

$$= -2x\frac{\cos n\pi x}{n\pi}\Bigg]_0^1 + \int_0^1 2\frac{\cos n\pi x}{n\pi}\,dx$$

$$= -2x\frac{\cos n\pi x}{n\pi}\Bigg]_0^1 + 2\frac{\sin n\pi x}{(n\pi)^2}\Bigg]_0^1 = -2\frac{\cos n\pi - 1}{f n\pi} = \frac{2\{1-(-1)^n\}}{n\pi}.$$

Therefore the Fourier series of is

$$f(x) \approx \sum_{n=1}^{\infty}\left\{b_n \sin\left(\frac{n\pi x}{L}\right)\right\} = \sum_{n=1}^{\infty}\left\{\frac{2\{1-(-1)^n\}}{n\pi}\sin(n\pi x)\right\}.$$

(1) By extending f out $(0,1)$ as a periodic function with period $L=1$, hence we have

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx = 2\int_0^1 x\ dx = 1,$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2n\pi x}{L}\right)dx = 2\int_0^1 x\ \cos(2n\pi x)\,dx$$

$$= x\frac{\sin 2n\pi x}{n\pi}\Bigg]_0^1 + \frac{\cos 2n\pi x}{2(n\pi)^2}\Bigg]_0^1 = 0,$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2n\pi x}{L}\right)dx = 2\int_0^1 x\ \sin(2n\pi x)\,dx$$

$$= -x\frac{\cos 2n\pi x}{n\pi}\Bigg]_0^1 - \frac{\sin 2n\pi x}{2(n\pi)^2}\Bigg]_0^1 = \frac{-1}{n\pi}.$$

Therefore the Fourier series of f is

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\cos\left(\frac{2n\pi x}{L}\right) + b_n\sin\left(\frac{2n\pi x}{L}\right)\right\} = \frac{1}{2} - \sum_{n=1}^{\infty}\left\{\frac{1}{n\pi}\sin\left(2n\pi x\right)\right\}.$$