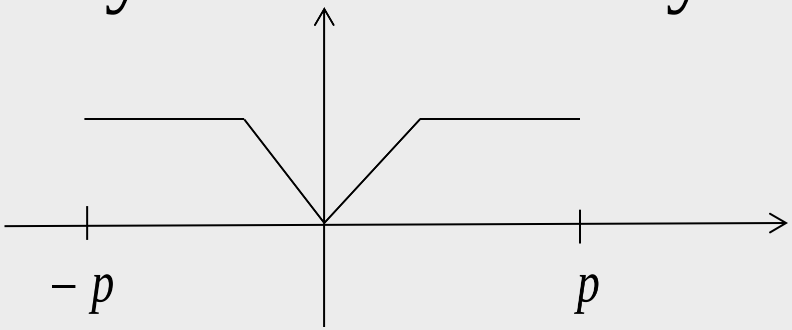
Fourier series of even and odd functions

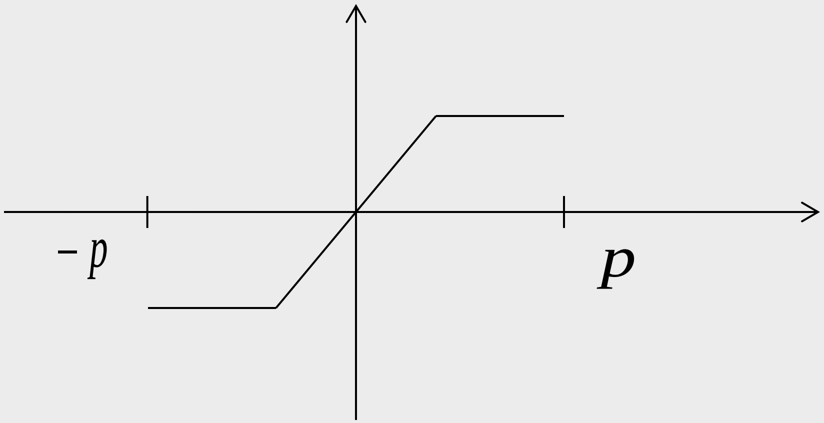
A function f is said to be even if $f(-x) = f(x)$ for all x in domain f. The graph of an even function is symmetric in the y-axis.



For example, the following functions are even functions:

$$f(x) = \cos(x), \quad f(x) = |x|, \quad f(x) = x^2.$$

A function f is said to be odd if $f(-x) = -f(x)$ for all x in domain f. The graph of an odd function is symmetric in the origin.



For example, the following functions are odd functions: $f(x) = \sin(x)$,

$f(x) = x$,

$f(x) = x(x^2 - |x|)$.

Remarks

Even function$\times$ even function =even function,

odd function $\times$ odd function = even function,

even function $\times$ odd function = odd function.

If f is an even function, then $\int_{-p}^{p} f(x)\, dx = 2\int_{0}^{p} f(x)\, dx.$

If f is an odd function, then $\int_{-p}^{p} f(x)\, dx = 0.$

The Fourier series of an even function f on the interval $(-p, p)$ is the cosine series

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{p}\right) \right\},$$

where

$$a_0 = \frac{2}{p} \int_0^p f(x)\,dx,$$

$$a_n = \frac{2}{p} \int_0^p f(x) \cos\left(\frac{n\pi x}{p}\right) dx.$$

The Fourier series of an odd function f on the interval $(-p, p)$ is the sine series

$$f(x) \approx \sum_{n=1}^{\infty}\left\{b_n \sin\left(\frac{n\pi x}{p}\right)\right\},$$

where

$$b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)dx.$$

Example. Let $f(x) = |x|, \quad -\pi < x < \pi,$ and satisfies $f(x+2\pi) = f(x) \quad for \ x \in R.$ Expand f in Fourier series

Solution. f is even, since $f(-x) = |-x| = |x| = f(x).$

Hence, the Fourier series of f is the cosine series

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{p} \right) \right\},$$

Where $a_0 = \frac{2}{p} \int_0^p f(x)\,dx, \;\; a_n = \frac{2}{p} \int_0^p f(x) \cos\left(\frac{n\pi x}{p} \right) dx.$

Here $(-p, p) = (-\pi, \pi) \Rightarrow p = \pi.$

Hence $a_0 = \frac{2}{p} \int_0^p f(x)\,dx = \frac{2}{\pi} \int_0^{\pi} |\,x\,|\,dx = \frac{2}{\pi} \int_0^{\pi} x\;dx = \pi,$

$$a_n = \frac{2}{p} \int_0^p f(x) \cos(\tfrac{n\pi x}{p})\,dx = \frac{2}{\pi} \int_0^{\pi} |\,x\,| \cos(nx)\,dx = \frac{2}{\pi} \int_0^{\pi} x\;\cos(nx)\,dx$$

$$= \frac{2}{\pi} \left[x \frac{\sin(nx)}{n} \Big|_0^{\pi} - \int_0^{\pi} \frac{\sin(nx)}{n}\,dx \right] = \tfrac{2}{\pi} \frac{\cos(nx)}{n^2} \Big]_0^{\pi} = \frac{2\{(-1)^n - 1\}}{n^2 \pi}.$$

Therefore the Fourier series of f is

$$f(x) \approx \frac{\pi}{2} + \sum_{n=1}^{\infty} \left[\frac{2\{(-1)^n - 1\}}{n^2 \pi} \cos(nx) \right].$$

Example. Let $f(x) = x, \quad -1 < x < 1,$ and satisfies $f(x+2) = f(x) \quad for\ all\ x \in R.$ Expand f in a Fourier series.

Solution. f is odd, since $f(-x) = -x = -f(x).$

Hence, the Fourier series of f is the sine series

$$f(x) \approx \sum_{n=1}^{\infty} \left\{ b_n \sin\left(\frac{n\pi x}{p}\right) \right\},$$

where $b_n = \frac{2}{p}\int_0^p f(x) \sin\left(\frac{n\pi x}{p}\right) dx.$

Here $(-p, p) = (-1, 1) \Rightarrow p = 1.$

Hence

$$b_n = \frac{2}{p}\int_0^p f(x)\sin(\tfrac{n\pi x}{p})\,dx = 2\int_0^1 x\sin(n\pi x)\,dx$$

$$= 2\left[-x\,\frac{\cos(n\pi x)}{n\pi}\Big]_0^1 + \int_0^1 \frac{\cos(n\pi x)}{n\pi}\,dx\right] = \frac{2(-1)^{n+1}}{n\pi}.$$

Therefore, the Fourier series of f is

$$f(x) \approx \sum_{n=1}^{\infty}\left[\frac{2(-1)^{n+1}}{n\pi}\sin\left(n\pi x\right)\right].$$