Fourier series

Let f be defined on $(-p, p)$ and satisfy $f(x+2p) = f(x)$ for all $x \in R$, then the Fourier series of f on the interval $(-p, p)$ is given by

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right) \right\},$$

where

$$a_0 = \frac{1}{p} \int_{-p}^{p} f(x)\,dx,$$

$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos\left(\frac{n\pi x}{p}\right) dx$$

$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin\left(\frac{n\pi x}{p}\right) dx$$

Theorem. Let f and f' be piecewise continuous on the interval $(-p, p)$. Then, the Fourier series of f on $(-p, p)$ converges to $f(x_0)$ if f is continuous at x_0 in $(-p, p)$, and converges to $\frac{1}{2}\left[f(x_0^{+}) + f(x_0^{-})\right]$ if is discontinuous at x_0, where $f(x_0^{+})$, *and* $f(x_0^{-})$ represent the RHL and LHL of f at x_0.

Example. Let

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ \pi - x, & 0 < x < \pi, \end{cases}$$

and satisfies

$$f(x + 2\pi) = f(x) \quad for\ all\ x \in R.$$

Find the Fourier series of f.

Solution.

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\,dx = \frac{\pi}{2}.$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi x}{p}\right)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\cos(nx)\,dx$$

$$\Rightarrow a_n = \frac{-\cos n\pi + 1}{n^2\pi} = \frac{1-(-1)^n}{n^2\pi},$$

and

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi x}{p}\right)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\sin(nx)dx = \tfrac{1}{n}.$$

Therefore the Fourier series of f is

$$f(x) \approx \frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{ \tfrac{1-(-1)^n}{n^2\pi}\cos(nx) + \tfrac{1}{n}\sin(nx) \right\}.$$

Now, the function f is discontinuous at $x = 0$, hence the series on the R.H.S. converges to $\frac{0+\pi}{2} = \frac{\pi}{2}$ at $x = 0$, thus

$$\frac{\pi}{2} = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\tfrac{1-(-1)^n}{n^2\pi} \right] \Rightarrow \frac{\pi}{4} = \sum_{n=1}^{\infty}\left[\tfrac{1-(-1)^n}{n^2\pi} \right].$$

Example. Find the Fourier series of $f(x) = \pi + x, \quad -\pi < x < \pi,$

then deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots.$

Solution. $a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi + x)\,dx = 2\pi.$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi x}{p}\right)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi + x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[(\pi + x)\frac{\sin nx}{n}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{\sin nx}{n}\,dx\right] = 0,$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi x}{p}\right)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi + x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[-(\pi + x)\frac{\cos nx}{n}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos nx}{n}\,dx\right] = \frac{2(-1)^{n+1}}{n}.$$

Hence the Fourier series of f is

$$f(x) \approx \pi + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2\pi} \sin(nx) .$$

Now, the function f is continuous at $x = \frac{\pi}{2}$, hence the series on the R.H.S. converges to $f(\frac{\pi}{2}) = \frac{3\pi}{2}$ at $x = \frac{\pi}{2}$, thus

$$\frac{3\pi}{2} = \pi + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(\tfrac{n\pi}{2}) \Rightarrow \frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(\tfrac{n\pi}{2}),$$

or $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$$