Orthogonal Sets of Functions

The inner product of two functions f and g on an interval $[a,b]$ is given by

$$(f,g)=\int_a^b f(x)g(x)\,dx.$$

The norm of a function f on the interval $[a,b]$ is given by

$$\|f\|=\sqrt{(f,f)}=\sqrt{\int_a^b f^2(x)\ dx}.$$

Orthogonal Functions

Two functions f and g are said to be orthogonal on the interval $[a,b]$ if

$$(f,g)=\int_a^b f(x)g(x)\,dx=0.$$

For example $f(x)=x$ and $g(x)=x^2$ are orthogonal on $[-1,1]$.

Since $(f,g)=\int_{-1}^{1} f(x)g(x)\,dx=\int_{-1}^{1} x^3\,dx=0$

Moreover, $\|f\|=\sqrt{(f,f)}=\sqrt{\int_{-1}^{1} x^2\,dx}=\sqrt{\dfrac{2}{3}}$

Orthogonal Sets

A set of functions $\{\phi_1(x), \phi_2(x), \ldots, \phi_n(x), \ldots\}$ is said to be orthogonal on the interval $[a, b]$ if

$$(\phi_m, \phi_n) = \int_a^b \phi_m(x)\, \phi_n(x)\, dx = 0 \quad for\ all\ \ m \neq n$$

If in addition then this set is called an orthonormal set.

$$\| \phi_n \| = 1 \quad for\ \ all\ \ n = 1,2,\ldots,$$

For example $\{\sin x, \sin 2x, \sin 3x, \ldots\}$ is an orthogonal set on $[0, \pi]$, because $(\sin(mx), \sin(nx)) = \int_0^\pi \sin(mx) \sin(nx)\, dx = 0,$

and

$$\| \sin mx \| = = \sqrt{\int_0^\pi (\sin mx)^2\, dx} = \sqrt{\frac{\pi}{2}} \quad for\ all\ \ m = 1,2,\ldots$$

Hence the corresponding orthonormal set is

$$\left\{ \sqrt{\frac{2}{\pi}}\, \phi_1(x),\ \sqrt{\frac{2}{\pi}}\, \phi_2(x), \ldots, \sqrt{\frac{2}{\pi}}\, \phi_n(x), \ldots \right\}.$$

Example 1

Show that the set of functions $\left\{1,\ \cos\left(\frac{n\pi x}{p}\right)\right\}, n = 1,2,\ldots$ is orthogonal on the interval $[0, p]$, and find the corresponding orthonormal sel.

Let $\phi_m(x) = \cos\left(\frac{n\pi x}{p}\right)$, then we have

$$(\phi_m(x),1) = \int_0^p \cos\left(\frac{m\pi x}{p}\right) dx = \frac{p}{m\pi}\sin\left(\frac{m\pi x}{p}\right)\Bigg]_0^p = 0.$$

and

$$(\phi_m(x),\phi_n(x)) = \int_0^p \cos\left(\frac{m\pi x}{p}\right)\cos\left(\frac{n\pi x}{p}\right) dx$$

$$= \int_0^p \frac{1}{2}\left\{\cos\left(\frac{(m+n)\pi x}{p}\right) + \cos\left(\frac{(m-n)\pi x}{p}\right)\right\} dx = 0.$$

Now,

$$\|1\| = \sqrt{\int_0^p 1\,dx} = \sqrt{p}\ ,\ and\ \ \|\phi_m\| = \sqrt{\int_0^p \cos^2\left(\frac{m\pi x}{p}\right)dx} = \sqrt{\frac{p}{2}}\ ,\ for\ all\ m \geq 1,$$

hence the corresponding orthonormal set is

$$\left\{\frac{1}{\sqrt{p}}, \sqrt{\frac{2}{p}}\cos\left(\frac{n\pi x}{p}\right)\right\}, n = 1,2,....$$

<u>Remark.</u>

$$\cos x \cos y = \tfrac{1}{2}\{\cos(x+y) + \cos(x-y)\},$$

$$\sin x \sin y = \tfrac{-1}{2}\{\cos(x+y) - \cos(x-y)\},$$

$$\cos x \sin y = \tfrac{1}{2}\{\sin(x+y) + \cos(y-x)\}.$$

Example 2

Show that the set of functions $\left\{1,\ \cos\left(\frac{m\pi x}{p}\right), \sin\left(\frac{n\pi x}{p}\right)\right\},\ m,n=1,2,\ldots$

is orthogonal on the interval $[-p,p]$.

Let $\phi_m(x)=\cos\left(\frac{m\pi x}{p}\right),\ \varphi_n(x)=\sin\left(\frac{n\pi x}{p}\right)$, then for $m \neq n$ we have

$$(\phi_m(x),\phi_n(x))=\int_{-p}^{p}\cos\left(\frac{m\pi x}{p}\right)\cos\left(\frac{n\pi x}{p}\right)dx$$

$$=\int_{-p}^{p}\frac{1}{2}\left\{\cos\left(\frac{(m+n)\pi x}{p}\right)+\cos\left(\frac{(m-n)\pi x}{p}\right)\right\}dx=0$$

Similarly,

$$(\varphi_m(x),\varphi_n(x))=\int_{-p}^{p}\sin\left(\frac{m\pi x}{p}\right)\sin\left(\frac{n\pi x}{p}\right)dx$$

$$=\int_{-p}^{p}\frac{-1}{2}\left\{\cos\left(\frac{(m+n)\pi x}{p}\right)-\cos\left(\frac{(m-n)\pi x}{p}\right)\right\}dx=0$$

$$(1, \phi_m(x)) = \int_{-p}^{p} \cos\left(\frac{m\pi x}{p}\right) dx = 0,$$

$$(1, \varphi_n(x)) = \int_{-p}^{p} \sin\left(\frac{m\pi x}{p}\right) dx = 0,$$

and

$$\begin{aligned}(\phi_m(x), \varphi_n(x)) &= \int_{-p}^{p} \cos\left(\frac{m\pi x}{p}\right) \sin\left(\frac{n\pi x}{p}\right) dx \\ &= \int_{-p}^{p} \frac{1}{2} \left\{ \sin\left(\frac{(m+n)\pi x}{p}\right) + \sin\left(\frac{(n-m)\pi x}{p}\right) \right\} dx \\ &= 0.\end{aligned}$$

Hence the set of functions is an orthogonal set.