$$ {}_{n|}\bar{A} = \int_n^\infty v^n \, {}_tp_x \, \mu_{x+t} \, dt $$

$s = t - n$

$ds = dt$

$$ = \int_n^\infty e^{-\delta t} \, {}_tp_x \, \mu_{x+t} \, dt $$

$$ = \int_0^\infty e^{-\delta(s+n)} \, {}_{s+n}p_x \, \mu_{s+n+x} \, ds $$

$$ = e^{-\delta n} \int_0^\infty e^{-\delta s} \, {}_np_x \, {}_sp_{x+n} \, \mu_{s+n+x} \, ds $$

$$ = e^{-\delta n} \, {}_np_x \int_0^\infty e^{-\delta s} \, {}_sp_{x+n} \, \underbrace{\mu_{s+n+x}}_{x+n} \, ds $$

$$ = e^{-\delta n} \, {}_np_x \, \bar{A}_{x+n} = {}_nE_x \, \bar{A}_{x+n} $$

$$ v^n \, {}_np_x = e^{-\delta n} \, {}_np_x = {}_nE_x $$

$$ \Rightarrow \boxed{ {}_{n|}\bar{A} = {}_nE_x \, \bar{A}_{x+n} } $$

$$ {}_{n|}\bar{A}_x = \bar{A}_x - \bar{A}^{1}_{x:\overline{n|}} $$

because

$$ {}_{n|}\bar{A}_x = \int_0^\infty v^n \, {}_tp_x \, \mu_{x+t} \, dt - \int_0^n v^t \, {}_tp_x \, \mu_{x+t} \, dt $$

$$ = \bar{A}_x - \bar{A}^{1}_{x:\overline{n|}} $$

$$ \boxed{ {}_{n|}\bar{A}_x = \bar{A}_x - \bar{A}^{1}_{x:\overline{n|}} } = \bar{A}_x - {}_nE_x \, \bar{A}_{x+n} $$

Also we can defined

(2)

$$ {}_{u}\bar{A}^{\,1}_{x:\overline{n}|} = \int_{u}^{u+n} e^{-\delta t}\ {}_{t}p_x\ \mu_{x+t}\,dt $$

where the present value random variable is actuarial (APV).

$$ Z = \begin{cases} 0 & \text{if } T_x < u \text{ or } T_x \geq u+n \\ e^{-\delta T_x} & \text{if } u \leq T_x \leq u+n \end{cases} $$

then

$$ {}_{u|}\bar{A}^{\,1}_{x:\overline{n}|} = \int_0^n e^{-\delta(s+u)}\ {}_{s+u}p_x\ \mu_{x+s+u}\,ds $$

$$ = e^{-\delta u}\ {}_{u}p_x \int_0^n e^{-\delta s}\ {}_{s}p_{x+u}\ \mu_{x+s+u}\,ds $$

$$ = e^{-\delta u}\ {}_{u}p_x\ \bar{A}^{\,1}_{x+u:\overline{n}|} $$

$$ = v^u\ {}_{u}p_x\ \bar{A}^{\,1}_{x+u:\overline{n}|} $$

$$ \boxed{{}_{u|}\bar{A}^{\,1}_{x:\overline{n}|} = {}_{n}E_x\ \bar{A}^{\,1}_{x+u:\overline{n}|}} $$

also

$$ {}_{u|}\bar{A}^{\,1}_{x:\overline{n}|} = \bar{A}^{\,1}_{x:\overline{u+n}|} - \bar{A}^{\,1}_{x:\overline{u}|} $$

becase

$$ \int_u^{u+n} e^{-\delta t}\ {}_{t}p_x\ \mu_{x+t}\,dt = \int_0^{u+n} e^{-\delta t}\ {}_{t}p_x\ \mu_{x+t}\,dt - \int_0^{u} e^{-\delta t}\ {}_{t}p_x\ \mu_{x+t}\,dt $$

$$ = \bar{A}^{\,1}_{x:\overline{u+n}|} - \bar{A}^{\,1}_{x:\overline{n}|} $$

③ $\rightarrow$ continunously increasing n-year endowment

$$Z = \begin{cases} T_x v^{T_x} & ; \quad T_x \leq n \\ n v^n & ; \quad T_x > n \end{cases}$$

$$\Rightarrow (\bar{I}\bar{A})_{x:\overline{n}|} = \int_0^n t v^t \, {}_tP_x \, \mu_{x+t} \, dt + \int_n^\omega n v^n \, {}_tP_x \, \mu_{x+t} \, dt$$

$$= (\bar{I}\bar{A})^1_{x:\overline{n}|} + n \int_n^\omega v^n \, {}_tP_x \, \mu_{x+t} \, dt$$

$$= (\bar{I}\bar{A})^1_{x:\overline{n}|} + n \, A_{x:\overline{n}|}^{\;\;1}$$

$$\int_n^\omega v^n \, {}_tP_x \, \mu_{x+t} \, dt = e^{-\delta n} \int_n^\infty {}_tP_x \, \mu_{x+t} \, dt = e^{-\delta n} Pr(T_x >$$

$$= e^{-\delta n} \, {}_nP_x = A_{x:\overline{n}|}^{\;\;1}$$

$$v^n \, {}_nP_x = {}_nE_x$$

$$\Rightarrow \boxed{(\bar{I}\bar{A})_{x:\overline{n}|} = (\bar{I}\bar{A})^1_{x:\overline{n}|} + n \, A_{x:\overline{n}|}^{\;\;1}}$$

4 Annually increasing n-year endowment

$$(I\bar{A})_{x:\overline{n|}}$$

$$Z = \begin{cases} \lfloor T_x + 1 \rfloor v^{T_x} & ; \quad T_x \leq n \\ n v^n & ; \quad T_x > n \end{cases}$$

$$\Rightarrow \quad (I\bar{A})_{x:\overline{n|}} = \int_0^n \lfloor t+1 \rfloor v^t \, {}_tp_x \, \mu_{x+t} \, dt + \int_n^\infty n v^n \, {}_tp_x \, \mu_{x+t} \, dt$$

$$= \int_0^n \lfloor t+1 \rfloor v^t \, {}_tp_x \, \mu_{x+t} \, dt + n v^n \int_n^\infty {}_tp_x \, \mu_{x+t} \, dt$$

$$= (I\bar{A})^{1}_{x:\overline{n|}} + n v^n \Pr(T_x > n)$$

$$= (I\bar{A})^{1}_{x:\overline{n|}} + n A_{x:\overset{1}{\overline{n|}}}$$

$$\boxed{(I\bar{A})_{x:\overline{n|}} = (I\bar{A})^{1}_{x:\overline{n|}} + n \, {}_nE_x}$$

Examples

You are given:

(i) The remaining lifetime for (x) is uniformly distributed over $[0, 2]$.

(ii) $\delta = 0,05$.

Calculate the following:

(a) $(\overline{I}\,\overline{A})_x$

(b) $(I\bar{A})_x$

(c) $(D\bar{A})^{1}_{x:\overline{2}|}$

Solution:

from (i) we have that:

$$f_x(t) = \frac{1}{2} \; ; \quad 0 \le t < 2$$

$$f_x(t) = {}_tp_x\,\mu_{x+t}$$

a/ under the policy of continuously increasing whole life benefit of 1SR for death at time t, we have

$$(\overline{I}\,\overline{A})_x = \int_0^2 t\,e^{-0,05t}\,\frac{1}{2}\,dt$$

$$\overset{\text{Integration by part}}{=} 0.9358$$

b/ under the policy of annually increasing whole life with a benefit of 1SR if death occurs within the first year, and 2SR if death occurs within the second year. We have;

$$(I\bar{A})_x = \int_0^1 e^{-0.05t}\frac{1}{2}dt + \int_1^2 2e^{-0.05t}\frac{1}{2}dt$$

$$= \left[-\frac{e^{-0.05t}}{0.1}\right]_0^1 + \left[-\frac{e^{-0.05t}}{0.05}\right]_1^2$$

$$= 0.48771 + 0.92784$$

$$= 1.4156.$$

c/ under the policy of annually decreasing n-year term with a benefit of 2SR if death occurs within the first year, and 1SR if death occurs within the second year. We have then:

$$(D\bar{A})^{1}_{x:\overline{2}|} = \int_0^1 2e^{-0.05t}\frac{1}{2}dt + \int_1^2 1e^{-0.05t}\frac{1}{2}dt$$

$$= 2(0.48771) + 0.5(0.92784)$$

$$= 1.4393.$$

To calculate the expected present value or the actuarial present value (APV) for term life insurance, deferred life insurance, and endowment insurance, we use the pure endowment, we must relate these types of insurance to the whole life insurance.

→ For example an n-year deferred life insurance on (x) is an n-year pure endowemt providing a whole life insurance on (x+n)

$$\boxed{{}_{n|}\bar{A}_x = {}_nE_x\ \bar{A}_{x+n}}$$ (continous).

→ An n-year term life insurance on (x) is a whole life insurance on (x) minus an n year deffered life insurance on (x)

$$\boxed{\bar{A}^{1}_{x:\overline{n|}} = \bar{A}_x - {}_nE_x\ \bar{A}_{x+n}}$$

→ An n-year endowment insurance on (x) is an n-year term insurance on (x) plus an n-year pure endowent on (x): $\bar{A}_{x:\overline{n|}} = \bar{A}^{1}_{x:\overline{n|}} + {}_nE_x$

$$= \bar{A}_x - {}_nE_x\ \bar{A}_{x+n} + {}_nE_x$$