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Mixed Wavelet Leaders Multifractal Formalism in a Product of Critical Besov Spaces

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Abstract. In this paper, we will prove (resp. study) the Baire generic validity of the upper-Hölder (resp. iso-Hölder) mixed wavelet leaders multifractal formalism on a product of two critical Besov spaces $B_{t_1}^{\frac{m}{t_1},q_1}(\mathbb{R}^m)\times B_{t_2}^{\frac{m}{t_2},q_2}(\mathbb{R}^m)$, for $t_1,t_2>0$, $q_1\le 1$ and $q_2\le 1$. Contrary to product spaces $B_{t_1}^{s_1,\infty}(\mathbb{R}^m)\times B_{t_2}^{s_2,\infty}(\mathbb{R}^m)$ with $s_1>\frac{m}{t_1}$ and $s_2>\frac{m}{t_2}$ (Ben Slimane in Mediterr J Math, 13(4):1513–1533, 2016) and $(B_{t_1}^{s_1,\infty}(\mathbb{R}^m)\cap C^{\gamma_1}(\mathbb{R}^m))\times(B_{t_2}^{s_2,\infty}(\mathbb{R}^m)\cap C^{\gamma_2}(\mathbb{R}^m)$ with $0<\gamma_1<s_1<\frac{m}{t_1}$ and $0<\gamma_2<s_2<\frac{m}{t_2}$ (Ben Abid et al. in Mediterr J Math, 13(6):5093–5118, 2016), all pairs of functions in the obtained generic set are not uniform Hölder. Nevertheless, the characterization of the upper bound of the Hölder exponent by decay conditions of local wavelet leaders suffices for our study.

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1. Introduction

Recently, many authors were interested in mixed multifractal spectra (see for example [1,3–6,14,30,31]).

In the framework of probability measures μ on $\mathbb{R}^m$, singularities are expressed by the pointwise exponent $h_\mu(x)$ of μ at x, given by

$$h_\mu(x)=\lim_{r\to 0}\frac{\log\mu(B(x,r))}{\log r}. \tag{1.1}$$

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Let dim denote the Hausdorff dimension. Conventionally $\dim\emptyset = -\infty$. The (single) spectrum of μ is given by

$$h \mapsto \dim E_\mu(h) \quad \text{where} \quad E_\mu(h) = \{x : h_\mu(x) = h\}. \tag{1.2}$$

The mixed multifractal spectrum of two measures μ_1 and μ_2 on $\mathbb{R}^m$ is given by

$$(h_1, h_2) \mapsto \dim E_{\mu_1}(h_1) \cap E_{\mu_2}(h_2). \tag{1.3}$$

It combines local characteristics which depend simultaneously on various different aspects of the underlying dynamical system, and allows to better understand the dynamics. Olsen [30] conjectured a mixed multifractal formalism which links the mixed spectrum (1.3) to the Legendre transform of mixed Rényi dimensions. Olsen obtained a general upper bound. He also proved that this bound is equality if both measures are selfsimilar with same contracting similarities.

In the framework of locally bounded functions $f : \mathbb{R}^m \to \mathbb{R}$, singularities are expressed by the Hölder exponent $h_f(x)$ of f at x, given by

$$h_f(x) = \sup\{h > 0 : f \in C^h(x)\}. \tag{1.4}$$

Recall that $f \in C^h(x)$, for h positive non-integer, means that

$$|f(y) - P(y - x)| \le C|y - x|^h \tag{1.5}$$

holds for all y in a neighborhood of x, for a constant C and a polynomial P of degree less than h.

Single spectra are described by either iso-Hölder spectrum (initially introduced by [19] in turbulence)

$$h \mapsto \dim E_f(h), \quad \text{where} \quad E_f(h) = \{x : h_f(x) = h\} \tag{1.6}$$

or upper-Hölder spectrum

$$h \mapsto \dim E_f^h, \quad \text{where} \quad E_f^h = \{x : h_f(x) \le h\}. \tag{1.7}$$

The mixed multifractal spectra of two functions f_1 and f_2 on $\mathbb{R}^m$ are given by

$$(h_1, h_2) \mapsto \dim E_{f_1}(h_1) \cap E_{f_2}(h_2) \tag{1.8}$$

and

$$(h_1, h_2) \mapsto \dim E_{f_1}^{h_1} \cap E_{f_2}^{h_2}. \tag{1.9}$$

The definitions can be extended for simultaneous Hölder exponents of finitely many functions.
Clearly

$$\dim E_{f_1}(h_1) \cap E_{f_2}(h_2) \le \min(\dim E_{f_1}(h_1), \quad \dim E_{f_2}(h_2)) \tag{1.10}$$

and

$$\dim E_{f_1}^{h_1} \cap E_{f_2}^{h_2} \le \min\left(\dim E_{f_1}^{h_1}, \dim E_{f_2}^{h_2}\right). \tag{1.11}$$

Note that if μ is a probability measure on $\mathbb{R}$ and f_μ is its primitive, then

$$h_{f_\mu}(x) = \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}, \tag{1.12}$$

when the right-hand term in (1.12) is less then 1.

In [1], the authors conjectured a mixed wavelet multifractal formalism which links the mixed spectrum (1.8) to the Legendre transform of a scaling function on the simultaneous continuous wavelet transforms of f_1 and f_2. They also proved the validity of that conjecture for pairs of selfsimilar functions with same contracting similarities. In [3], the authors extended the validity for pairs of some non-selfsimilar functions.

In [14], the second author of this paper conjectured a mixed wavelet leaders multifractal formalism which involves a mixed wavelet leaders scaling function $\omega_{(f_1,f_2)}(p_1,p_2)$. He also proved that, Baire generically, the upper bound (1.11) becomes equality and $\dim E_{f_1}^{h_1} \cap E_{f_2}^{h_2}$ coincides with the Legendre transform of $\omega_{(f_1,f_2)}(p_1,p_2)$, for pairs (f_1,f_2) in a product of continuous Besov spaces $B_{t_1}^{s_1,\infty}(\mathbb{R}^m) \times B_{t_2}^{s_2,\infty}(\mathbb{R}^m)$, for $s_1 > \frac{m}{t_1}$ and $s_2 > \frac{m}{t_2}$. In [6], this result was extended in a product of intersections of a non-continuous Besov space with a Hölder space $(B_{t_1}^{s_1,\infty}(\mathbb{R}^m) \cap C^{\gamma_1}(\mathbb{R}^m)) \times (B_{t_2}^{s_2,\infty}(\mathbb{R}^m) \cap C^{\gamma_2}(\mathbb{R}^m)$, for $0 < \gamma_1 < s_1 < \frac{m}{t_1}$ and $0 < \gamma_2 < s_2 < \frac{m}{t_2}$. The Baire equality of (1.10) on these spaces was also studied in [6,14]. To achieve the results, the authors have used the wavelet leaders characterization of the Hölder exponent (1.4) of a uniform Hölder function.

In this paper, we will prove (resp. study) the Baire generic validity of the upper-Hölder (resp. iso-Hölder) mixed wavelet leaders multifractal formalism on a product of two critical Besov spaces $B_{t_1}^{\frac{m}{t_1},q_1}(\mathbb{R}^m) \times B_{t_2}^{\frac{m}{t_2},q_2}(\mathbb{R}^m)$, for $t_1, t_2 > 0$, $q_1 \leq 1$ and $q_2 \leq 1$. Contrary to the above product spaces, all pairs of functions in the obtained generic set are not uniform Hölder. Nevertheless, the characterization of the upper bound of the Hölder exponent by decay conditions of local wavelet leaders suffices for our study.

Ideas of this paper together with [6,14] allow to cover the case of any finite product of above Besov spaces.

Note that, Jaffard and Meyer [27] proved that if $q > 1$ then functions in $B_t^{\frac{m}{t},q}(\mathbb{R}^m)$ are not necessarily locally bounded. They also computed the single Hölder spectrum generically in $B_t^{\frac{m}{t},q}(\mathbb{R}^m)$ if $0 < q \leq 1$. In the case where $0 < t < q \leq 1$, in order to simplify the notations, the generic set was constructed in the case where $m = 1$. In this paper, we clarify and give the construction for any m.

Note that the multifractal formalism of infinitely simultaneous many pointwise singularities was studied by Peyrière [31]. Its validity holds under some Frostman assumption. The check of this assumption proves to be very difficult.

Note also that iso-Hölder spectrum and multifractal formalism of single functions have been studied under selfsimilarity assumptions on f [2,7–12,15,21], or for a class of particular random processes [22], or for specific functions f [13,20], or even generically in either Baire sense [24,27,28] or prevalence sense [17,18].

In the next section, we will recall the statement of the mixed wavelet leaders multifractal formalisms and summarize our main results.
In Sects. 3, 4 and 5, we give the proofs.

2. Mixed Wavelet Leaders Multifractal Formalisms and Main Results

2.1. Mixed Wavelet Leaders Multifractal Formalism

Let $\{2^{\frac{mj}{2}}\psi^r(2^jx-k),\ r=1,\dots,2^m-1, j\geq 0,\ k\in\mathbb{Z}^m\}\cup\{\phi(x-k),\ k\in\mathbb{Z}^m\}$ form an orthonormal wavelet basis of $L^2(\mathbb{R}^m)$ in the Schwartz class (see [29]). We will omit the index r. Using the notation $\lambda=\lambda_{j,k}=k2^{-j}+[0,2^{-j})^m$ and $\psi_\lambda(x)=\psi(2^jx-k)$, the wavelet coefficients $c_k(f)$ and $C_\lambda(f)$ of a function in $L^2(\mathbb{R}^m)$ are given by

$$c_k(f)=\int_{\mathbb{R}^m}\phi(t-k)f(t)\mathrm{d}t \tag{2.1}$$

and

$$C_\lambda(f)=2^{mj}\int_{\mathbb{R}^m}\psi_\lambda(t)f(t)\mathrm{d}t. \tag{2.2}$$

The usual modification holds for (2.1) and (2.2) when f is a tempered distribution.

In [29], it is proved that any function f in $L^2(\mathbb{R}^m)$ can be expanded as

$$f(x)=\sum_{k\in\mathbb{Z}^m}c_k(f)\phi(x-k)+\sum_{j\geq 0}\sum_{\lambda\in\Lambda_j}C_\lambda(f)\psi_\lambda(x), \tag{2.3}$$

where

$$\Lambda_j=\{\lambda_{j,k}:k\in\mathbb{Z}^m\}. \tag{2.4}$$

Recall that $B_t^{s,q}(\mathbb{R}^m)$ is the space of all functions f satisfying

$$||f||:=\left(\sum_{k\in\mathbb{Z}^m}|c_k(f)|^t\right)^{1/t}+\left(\sum_{j\geq 0}\left(\sum_{\lambda\in\Lambda_j}|C_\lambda(f)2^{(s-\frac{m}{t})j}|^t\right)^{q/t}\right)^{1/q}<\infty \tag{2.5}$$

(with the usual modification when $t=\infty$ and/or $q=\infty$).

Let $f\in B_\infty^{0,\infty}(\mathbb{R}^m)$, there exists $C>0$ such that

$$\forall\ j\geq 0\ \ \forall\ \lambda\in\Lambda_j\quad |C_\lambda(f)|\leq C. \tag{2.6}$$

For $\lambda\in\Lambda_j$ let

$$d_\lambda(f)=\sup_{\lambda'\subset\lambda}|C_\lambda(f)| \tag{2.7}$$

denote the wavelet leader coefficient of f in the cube λ.

Let $x\in\mathbb{R}^m$ and $j\geq 0$. Denote by $\lambda_{j,k(x)}$ the unique dyadic cube in Λ_j that contains x. Put

$$Adj(\lambda_{j,k(x)})=\prod_{i=1}^m[(k_i(x)-1)2^{-j},(k_i(x)+2)2^{-j}). \tag{2.8}$$

Clearly $Adj(\lambda_{j,k(x)})$ is the union of $\lambda_{j,k(x)}$ and its 3^m-1 adjacent cubes in Λ_j.

The local wavelet leader around x at scale j is defined by

$$d_j(f)(x)=\sup_{\lambda'\subset Adj(\lambda_{j,k(x)})}|C_{\lambda'}(f)|. \tag{2.9}$$

If f is uniform Hölder, then the Hölder exponent $h_f(x)$ given in (1.4) is characterized by a decay condition of the wavelet leaders near x (see [25])

$$h_f(x) = \liminf_{j\to\infty} \frac{\log d_j(f)(x)}{\log 2^{-j}}. \tag{2.10}$$

Recall that f is uniform Hölder if there exists $\delta \in (0,1)$ and $C > 0$ such that

$$\forall\, x, y \in \mathbb{R}^m \quad |f(x) - f(y)| \le C|x-y|^{\delta}. \tag{2.11}$$

Without any assumption of uniform regularity on f we only have

$$h_f(x) \le \liminf_{j\to\infty} \frac{\log d_j(f)(x)}{\log 2^{-j}}. \tag{2.12}$$

If Ω is a bounded subset of $\mathbb{R}^m$ and $j \ge 0$, put

$$\Lambda_j(\Omega) = \{\lambda \in \Lambda_j : \lambda_j \subset \Omega\}. \tag{2.13}$$

Let f_1 and f_2 be two functions in $B^{0,\infty}_{\infty}(\mathbb{R}^m)$. The mixed wavelet leaders scaling function $\omega^{\Omega}_{(f_1,f_2)}(p_1,p_2)$ on Ω, for $p_1, p_2 > 0$, is defined by

$$\omega^{\Omega}_{(f_1,f_2)}(p_1,p_2) = \liminf_{j\to\infty} \frac{\log\left(2^{-mj} \sum_{\lambda\in\Lambda_j(\Omega)} ((d_\lambda(f_1))^{p_1}(d_\lambda(f_2))^{p_2})\right)}{\log(2^{-j})}. \tag{2.14}$$

The mixed wavelet leaders scaling function $\omega_{(f_1,f_2)}(p_1,p_2)$, for $p_1, p_2 > 0$, is defined by

$$\omega_{(f_1,f_2)}(p_1,p_2) = \inf_{\Omega} \omega^{\Omega}_{(f_1,f_2)}(p_1,p_2). \tag{2.15}$$

The Legendre transform of the function $\omega_{(f_1,f_2)}$ is defined by

$$\omega^*_{(f_1,f_2)}(h_1,h_2) = \inf_{p_1>0,\ p_2>0} (h_1p_1 + h_2p_2 - \omega_{(f_1,f_2)}(p_1,p_2)). \tag{2.16}$$

The iso-Hölder mixed wavelet leaders multifractal formalism (see [14]) states that

$$\dim E_{f_1}(h_1) \cap E_{f_2}(h_2) = m + \omega^*_{(f_1,f_2)}(h_1,h_2). \tag{2.17}$$

The upper-Hölder mixed wavelet leaders multifractal formalism (see [14]) states that

$$\dim E^{h_1}_{f_1} \cap E^{h_2}_{f_2} = m + \omega^*_{(f_1,f_2)}(h_1,h_2). \tag{2.18}$$

Remark 1. Besov spaces $B^{s,q}_t(\mathbb{R}^m)$, $s,t,q > 0$, are Baire spaces. If moreover $s \ge m/t$ then $B^{s,q}_t(\mathbb{R}^m)$ is included in $B^{0,\infty}_{\infty}(\mathbb{R}^m)$.

Recall that in a Baire space E any countable intersection of open dense sets is dense and called a G_δ-set or residual set. Moreover, if a property (P) on E holds on a G_δ-set, (P) holds Baire generically in E.

In [14], the second author proved that, Baire generically the upper-Hölder mixed wavelet leaders multifractal formalism holds for pairs (f_1, f_2) in a product of continuous Besov spaces $B^{s_1,\infty}_{t_1}(\mathbb{R}^m) \times B^{s_2,\infty}_{t_2}(\mathbb{R}^m)$, for $s_1 > \frac{m}{t_1}$ and $s_2 > \frac{m}{t_2}$. In [6], this result was extended in a product of intersections of a non-continuous Besov space with a Hölder space $(B^{s_1,\infty}_{t_1}(\mathbb{R}^m)\cap C^{\gamma_1}(\mathbb{R}^m))\times(B^{s_2,\infty}_{t_2}(\mathbb{R}^m)\cap C^{\gamma_2}(\mathbb{R}^m))$, for $0<\gamma_1<s_1<\frac{m}{t_1}$ and $0<\gamma_2<s_2<\frac{m}{t_2}$. The Baire validity of the iso-Hölder mixed wavelet leaders multifractal formalism was also studied in [6,14]. To achieve the results, the authors have used the wavelet characterization (2.10) of the Hölder exponent of a uniform Hölder function.

In this paper, we will prove (resp. study) the Baire generic validity of the upper-Hölder (resp. iso-Hölder) mixed wavelet leaders multifractal formalism on a product of two critical Besov spaces $B^{\frac{m}{t_1},q_1}_{t_1}(\mathbb{R}^m)\times B^{\frac{m}{t_2},q_2}_{t_2}(\mathbb{R}^m)$, for $q_1\le 1$ and $q_2\le 1$. Contrary to the above spaces, functions in critical Besov spaces are not necessarily uniform Hölder. Bound (2.12) can be applied, but not (2.10). Note that, for $q_1>1$ and $q_2>1$, functions of these spaces are not necessarily locally bounded (see [27]).

From now on, we will not write $(\mathbb{R}^m)$ in $B^{\frac{m}{t_i},q_i}_{t_i}(\mathbb{R}^m)$. For $L=(\ell_1,\dots,\ell_m)\in\mathbb{Z}^m$, let $\mathcal{C}_L$ be the cube $L+[0,1]^m$ of $\mathbb{R}^m$. Our main results are summarized in the following theorems.

Theorem 2.1. *Let $q_1,q_2,t_1,t_2>0$. Let $b_i=\max\{q_i,t_i\}$, $i\in\{1,2\}$. Then, for all $(f_1,f_2)\in B^{\frac{m}{t_1},q_1}_{t_1}\times B^{\frac{m}{t_2},q_2}_{t_2}$*

1.

$$\omega_{(f_1,f_2)}(p_1,p_2)\ge\begin{cases} m\left(\frac{p_1}{b_1}+\frac{p_2}{b_2}\right) & \text{if } \frac{p_1}{b_1}+\frac{p_2}{b_2}<1\\ m & \text{if } \frac{p_1}{b_1}+\frac{p_2}{b_2}\ge 1\end{cases}\tag{2.19}$$

2.

$$m+\omega^*_{(f_1,f_2)}(h_1,h_2)\begin{cases} =-\infty & \text{if } h_1<0 \text{ or } h_2<0\\ \le\min\{h_1b_1,h_2b_2,m\} & \text{else.}\end{cases}\tag{2.20}$$

Theorem 2.2. *Let $q_1,q_2,t_1,t_2>0$. Let $b_i=\max\{q_i,t_i\}$, $i\in\{1,2\}$. Then, Baire generically, pairs of functions (f_1,f_2) in $B^{\frac{m}{t_1},q_1}_{t_1}\times B^{\frac{m}{t_2},q_2}_{t_2}$ satisfy for all $L\in\mathbb{Z}^m$*

$$\omega^{\mathcal{C}_L}_{(f_1,f_2)}(p_1,p_2)=\omega_{(f_1,f_2)}(p_1,p_2)=\begin{cases} m\left(\frac{p_1}{b_1}+\frac{p_2}{b_2}\right) & \text{if } \frac{p_1}{b_1}+\frac{p_2}{b_2}<1\\ m & \text{if } \frac{p_1}{b_1}+\frac{p_2}{b_2}\ge 1\end{cases}\tag{2.21}$$

and

$$m+\omega^*_{(f_1,f_2)}(h_1,h_2)=\begin{cases} -\infty & \text{if } h_1<0 \text{ or } h_2<0\\ \min\{h_1b_1,h_2b_2,m\} & \text{else.}\end{cases}\tag{2.22}$$

Theorem 2.3. *Let $q_1,q_2\le 1$ and $t_1,t_2>0$. Set $b_i=\max\{q_i,t_i\}$, $i\in\{1,2\}$. Baire generically, pairs of functions (f_1,f_2) in $B^{\frac{m}{t_1},q_1}_{t_1}\times B^{\frac{m}{t_2},q_2}_{t_2}$ satisfy for all $L\in\mathbb{Z}^m$*

1.

$$\forall\,(h_1,h_2)\ \ dim\,E^{h_1}_{f_1}\cap E^{h_2}_{f_2}\cap\mathcal{C}_L=dim\,E^{h_1}_{f_1}\cap E^{h_2}_{f_2}=m+\omega^*_{(f_1,f_2)}(h_1,h_2).\tag{2.23}$$

2. *If* $h_i \notin [0, \frac{m}{b_i}]$ *for either* $i = 1$ *or 2 then*

$$dimE_{f_1}(h_1) \cap E_{f_2}(h_2) \cap \mathcal{C}_L = dimE_{f_1}(h_1) \cap E_{f_2}(h_2) = -\infty \tag{2.24}$$

$$\begin{cases} = m + \omega^*_{(f_1,f_2)}(h_1, h_2) \text{ if } h_1 < 0 or h_2 < 0 \\ < m + \omega^*_{(f_1,f_2)}(h_1, h_2) \text{ else.} \end{cases} \tag{2.25}$$

3. *If* $(h_1, h_2) \in [0, \frac{m}{b_1}] \times [0, \frac{m}{b_2}]$ *then*

$$dimE_{f_1}(h_1) \cap E_{f_2}(h_2) \cap \mathcal{C}_L = dimE_{f_1}(h_1) \cap E_{f_2}(h_2) \leq m + \omega^*_{(f_1,f_2)}(h_1, h_2). \tag{2.26}$$

If moreover $h_1 b_1 = h_2 b_2$ *then*

$$dimE_{f_1}(h_1) \cap E_{f_2}(h_2) \cap \mathcal{C}_L = dimE_{f_1}(h_1) \cap E_{f_2}(h_2) = m + \omega^*_{(f_1,f_2)}(h_1, h_2). \tag{2.27}$$

3. Proof of Theorem 2.1

Let (f_1, f_2) in $B_{t_1}^{\frac{m}{t_1},q_1} \times B_{t_2}^{\frac{m}{t_2},q_2}$. For $i \in \{1, 2\}$, put $b_i = \max\{q_i, t_i\}$.

1. If Ω is a bounded subset of $\mathbb{R}^m$ and $f \in B_\infty^{0,\infty}(\mathbb{R}^m)$, define the wavelet leaders scaling function $\omega_f^\Omega(p)$ on Ω, for $p > 0$, by

$$\omega_f^\Omega(p) = \liminf_{j\to\infty} \frac{\log\left(2^{-mj} \sum_{\lambda\in\Lambda_j(\Omega)} (d_\lambda(f))^p\right)}{\log(2^{-j})}, \tag{3.1}$$

where $\Lambda_j(\Omega)$ is in (2.13).

The wavelet leaders scaling function $\omega_f(p)$, for $p > 0$, is defined by (see [25])

$$\omega_f(p) = \inf_\Omega \omega_f^\Omega(p). \tag{3.2}$$

Remark 2. In [25], it is shown that $\omega_f(p)$ does not depend on the chosen sufficiently smooth wavelet basis.

By Hölder inequality, we have (see Proposition 3.1 of [14]) for all $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$

$$\omega_{(f_1,f_2)}(p_1, p_2) \geq \frac{1}{p}\omega_{f_1}(pp_1) + \frac{1}{q}\omega_{f_2}(qp_2), \tag{3.3}$$

Lemma 3.1. *If* $f \in B_t^{\frac{m}{t},q}$ *and* $b = \max\{q, t\}$ *then*

$$\forall\, p > 0 \quad \omega_f(p) \geq \min\left\{m, \frac{mp}{b}\right\}.$$

Proof. For $p > 0$ and $s > 0$, the oscillation space $O_p^s(\mathbb{R}^m)$ is defined by

$$f \in O_p^s(\mathbb{R}^m) \Longleftrightarrow f \in B_p^{s,\infty}(\mathbb{R}^m) \quad \text{and} \sup_{j\geq 0} \left(2^{(sp-m)j} \sum_{\lambda\in\Lambda_j} (d_\lambda(f))^p\right) < \infty. \tag{3.4}$$

Its local version $O^{s/p}_{p,\mathrm{loc}}(\mathbb{R}^m)$ is the space of functions f such that the restriction of f on any bounded open set Ω in $\mathbb{R}^m$ coincides with a function in $O^s_p(\mathbb{R}^m)$. In [25], it is also shown that

$$\omega_f(p) = \sup\{s : f \in O^{s/p}_{p,\mathrm{loc}}(\mathbb{R}^m)\}. \tag{3.5}$$

(a) Let $0 < q \le \min(1,t)$. Since $q \le t$, then $B^{m/t,q}_t \hookrightarrow B^{m/t,t}_t$. In Proposition 2 in [26], it is shown that $B^{m/t,t}_t \hookrightarrow O^{m/t}_t$.
- If $p \le t$ then $O^{m/t}_{t,\mathrm{loc}} \subset O^{m/t}_{p,loc}$. Thus $\omega_f(p) \ge mp/t$.
- If $p \ge t$ then in Proposition 2 in [26] it is shown that $O^{m/t}_t \hookrightarrow O^{m/p}_p$. Thus $\omega_f(p) \ge m$.

(b) Let $0 < t < q \le 1$ and $f \in B^{m/t,q}_t$.
- If $p \ge q$ then $p > t$. It is known that $B^{m/t,q}_t \hookrightarrow B^{m/p,q}_p$. Since $p \ge q$ then $B^{m/p,q}_p \hookrightarrow B^{m/p,p}_p$. In Proposition 2 in [26], it is shown that $B^{m/p,p}_p \hookrightarrow O^{m/p}_p$. Therefore $\omega_f(p) \ge m$. In particular, $\omega_f(q) \ge m$.
- Let now $p < q$. In [25], by Hölder's inequality, it is shown that, if both f and the wavelets are compactly supported then

$$\sum_{\lambda\in\Lambda_j} (d_\lambda(f))^q \ge C 2^{mj(1-\frac{q}{p})} \left(\sum_{\lambda\in\Lambda_j} (d_\lambda(f))^p\right)^{q/p}.$$

We deduce that

$$\omega_\Omega(q) \le \frac{q}{p}\omega_\Omega(p)$$

and so

$$\omega_f(q) \le \frac{q}{p}\omega_f(p).$$

It follows from Remark 2 that this property remains valid if the wavelets are sufficiently smooth.
Since $\omega_f(q) \ge m$, then

$$\omega_f(p) \ge \frac{mp}{q}.$$

□

Clearly

$$\min\left\{m, \frac{mp}{b}\right\} = \frac{m}{2}\left(1 + \frac{p}{b} - \left|1 - \frac{p}{b}\right|\right).$$

Thus for all $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$

$$\omega_{(f_1,f_2)}(p_1,p_2) \ge \frac{m}{2}\left(1 + \frac{p_1}{b_1} + \frac{p_2}{b_2} - \left|\frac{1}{p} - \frac{p_1}{b_1}\right| - \left|\frac{1}{q} - \frac{p_2}{b_2}\right|\right).$$

Put $x = \frac{1}{p}$. We have for all $x \in]0,1[$

$$\omega_{(f_1,f_2)}(p_1,p_2) \ge \frac{m}{2}\left(1 + \frac{p_1}{b_1} + \frac{p_2}{b_2} - \left|x - \frac{p_1}{b_1}\right| - \left|1 - x - \frac{p_2}{b_2}\right|\right).$$

So

$$\omega_{(f_1,f_2)}(p_1,p_2) \geq \frac{m}{2}\left(1+\frac{p_1}{b_1}+\frac{p_2}{b_2}-\inf_{x\in[0,1]}\varphi(x)\right),$$

where

$$\varphi(x)=\left|x-\frac{p_1}{b_1}\right|+\left|x-(1-\frac{p_2}{b_2})\right|.$$

The first assertion of the point can be deduced from the following lemma.

Lemma 3.2.

$$\inf_{x\in[0,1]}\varphi(x)=\left|1-\frac{p_2}{b_2}-\frac{p_1}{b_1}\right|=\begin{cases}1-\frac{p_2}{b_2}-\frac{p_1}{b_1} & \textit{if } \frac{p_2}{b_2}+\frac{p_1}{b_1}\leq 1\\ \frac{p_2}{b_2}+\frac{p_1}{b_1}-1 & \textit{if } \frac{p_2}{b_2}+\frac{p_1}{b_1}\geq 1.\end{cases}$$

Proof. Write $\varphi(x)=|x-a_1|+|x-a_2|$, where $a_1=\frac{p_1}{b_1}>0$, and $a_2=1-\frac{p_2}{b_2}<1$.

- If $a_1,a_2\in[0,1]$ and $a_1\leq a_2$, i.e., $\frac{p_1}{b_1}+\frac{p_2}{b_2}\leq 1$, then

$$\forall\, x\in[0,1]\quad \varphi(x)=\begin{cases}a_1+a_2-2x & \text{if } x\leq a_1\\ a_2-a_1 & \text{if } a_1\leq x\leq a_2\\ 2x-a_1-a_2 & \text{if } a_2\leq x.\end{cases}$$

  Thus

$$\inf_{x\in[0,1]}\varphi(x)=a_2-a_1=1-\frac{p_2}{b_2}-\frac{p_1}{b_1}.$$

- If $a_1,a_2\in[0,1]$ and $a_2\leq a_1$, i.e., $\frac{p_1}{b_1}\leq 1$, $\frac{p_2}{b_2}\leq 1$ and $\frac{p_1}{b_1}+\frac{p_2}{b_2}\geq 1$, then

$$\forall\, x\in[0,1]\quad \varphi(x)=\begin{cases}a_1+a_2-2x & \text{if } x\leq a_2\\ a_1-a_2 & \text{if } a_2\leq x\leq a_1\\ 2x-a_1-a_2 & \text{if } a_1\leq x.\end{cases}$$

  Thus

$$\inf_{x\in[0,1]}\varphi(x)=a_1-a_2=\frac{p_1}{b_1}+\frac{p_2}{b_2}-1.$$

- If $a_1\in[0,1]$ and $a_2<0$, i.e., $\frac{p_1}{b_1}\leq 1$, $\frac{p_2}{b_2}>1$, then

$$\forall\, x\in[0,1]\quad \varphi(x)=\begin{cases}a_1-a_2 & \text{if } x\leq a_1\\ 2x-a_1-a_2 & \text{if } a_1\leq x.\end{cases}$$

  Thus

$$\inf_{x\in[0,1]}\varphi(x)=a_1-a_2=\frac{p_1}{b_1}+\frac{p_2}{b_2}-1.$$

- If $a_1>1$ and $a_2\in[0,1]$, i.e., $\frac{p_1}{b_1}>1$, $\frac{p_2}{b_2}\leq 1$, then

$$\forall\, x\in[0,1]\quad \varphi(x)=\begin{cases}a_1+a_2-2x & \text{if } x\leq a_2\\ a_1-a_2 & \text{if } a_2\leq x.\end{cases}$$

  Thus

$$\inf_{x\in[0,1]}\varphi(x)=a_1-a_2=\frac{p_1}{b_1}+\frac{p_2}{b_2}-1.$$

- If $a_1 > 1$ and $a_2 < 0$, i.e., $\frac{p_1}{b_1} > 1$, $\frac{p_2}{b_2} > 1.$, then
$$\forall\, x \in [0,1] \quad \varphi(x) = a_1 - a_2.$$
Thus
$$\inf_{x\in[0,1]} \varphi(x) = a_1 - a_2 = \frac{p_1}{b_1} + \frac{p_2}{b_2} - 1.$$

□

For $p_1, p_2 > 0$, set
$$B(p_1,p_2) = \begin{cases} m\left(\frac{p_1}{b_1} + \frac{p_2}{b_2}\right) & \text{if } \frac{p_1}{b_1} + \frac{p_2}{b_2} < 1 \\ m & \text{if } \frac{p_1}{b_1} + \frac{p_2}{b_2} \geq 1. \end{cases}$$
From above, we deduce
$$\forall\, p_1, p_2 > 0 \quad \omega_{(f_1,f_2)}(p_1,p_2) \geq B(p_1,p_2).$$

2. The previous lower bound yields
$$\forall\, h_1, h_2 > 0 \quad m + \omega^*_{(f_1,f_2)}(h_1,h_2) \leq m + B^*(h_1,h_2).$$

Lemma 3.3. *We have*
$$m + B^*(h_1,h_2) = \begin{cases} -\infty & \textit{if } h_1 < 0 \textit{ or } h_2 < 0 \\ \min\{h_1b_1, h_2b_2, m\} & \textit{else.} \end{cases} \tag{3.6}$$

Proof. To compute $B^*(h_1,h_2) = \inf_{p_1,p_2>0}(h_1p_1 + h_2p_2 - B(p_1,p_2))$, we split $(0,+\infty)^2$ as
$$(0,+\infty)^2 = D_1 \cup D_2,$$
where
$$D_1 = \left\{(p_1,p_2) \in (0,+\infty)^2;\ \frac{p_1}{b_1} + \frac{p_2}{b_2} \leq 1\right\},$$
and
$$D_2 = \left\{(p_1,p_2) \in (0,+\infty)^2;\ \frac{p_1}{b_1} + \frac{p_2}{b_2} \geq 1\right\}.$$
Clearly
$$B^*(h_1,h_2) = \min\left\{\inf_{D_1} g, \inf_{D_2} g\right\}. \tag{3.7}$$
Clearly
$$\forall\, i = 1,2 \quad \inf_{D_i} g = \inf_{\partial D_i} g,$$
where ∂D_i is the boundary of D_i.

We have
$$\partial D_1 = \{(p,0);\ 0 \leq p \leq b_1\} \cup \left\{\left(p, -\frac{b_2}{b_1}p + b_2\right);\ 0 \leq p \leq b_1\right\} \cup \{(0,p);\ 0 \leq p \leq b_2\}$$

and

$$\partial D_2 = \left\{ \left(p, -\frac{b_2}{b_1} p + b_2 \right) ;\ 0 \le p \le b_1 \right\} \cup \{(0,p);\ p \ge b_2\} \cup \{(p,0);\ p \ge b_1\}.$$

Then

$$\inf_{\partial D_1} g = \min \left\{ \inf_{0 \le p \le b_1} g_1(p), \inf_{0 \le p \le b_2} g_2(p), \inf_{0 \le p \le b_1} g_3(p) \right\},$$
$$\inf_{\partial D_2} g = \min \left\{ \inf_{0 \le p \le b_1} g_3(p), \inf_{p \ge b_2} g_4(p), \inf_{p \ge b_1} g_5(p) \right\},$$

where

$$g_1(p) = (h_1 b_1 - m)\frac{p}{b_1},$$
$$g_2(p) = (h_2 b_2 - m)\frac{p}{b_2}, \quad g_3(p) = (h_1 b_1 - h_2 b_2)\frac{p}{b_1} + h_2 b_2 - m$$

$$g_4(p) = h_2 p - m, \quad g_5(p) = h_1 p - m.$$

Thus

$$B^*(h_1,h_2) = \min \left\{ \inf_{0 \le p \le b_1} g_1(p), \inf_{0 \le p \le b_2} g_2(p), \inf_{0 \le p \le b_1} g_3(p), \inf_{p \ge b_2} g_4(p), \inf_{p \ge b_1} g_5(p) \right\}$$
$$= \begin{cases} -\infty & \text{if } h_1 < 0 \text{ or } h_2 < 0 \\ \min\{h_1 b_1, h_2 b_2, m\} & \text{else.} \end{cases}$$

□

4. Proof of Theorem 2.2

4.1. Construction of a Saturating Pair (F_1, F_2)

We will first construct a pair (F_1, F_2) of functions that will satisfy (2.21).

For $L = (\ell_1, \dots, \ell_m) \in \mathbb{Z}^m$, put

$$|L| = |\ell_1| + \cdots + |\ell_m| \tag{4.1}$$

and $\mathcal{C}_L$ the cube $L + [0,1]^m$ of $\mathbb{R}^m$.

Let

$$F(x) = \sum_{L \in \mathbb{Z}^m} \sum_{j \ge 1} \sum_{\lambda \in \Lambda_j(\mathcal{C}_L)} C_\lambda(F) \psi_\lambda(x). \tag{4.2}$$

Remark 3. If $\lambda \in \Lambda_j(\mathcal{C}_0)$. Let $L \in \mathbb{Z}^m$ and $\widetilde{\lambda} = L + \lambda$ be the cube obtained from λ by the translation of L. Clearly $\widetilde{\lambda} \in \Lambda_j(\mathcal{C}_L)$. We will put

$$C_{\widetilde{\lambda}}(F) = 2^{-|L|} C_\lambda(F). \tag{4.3}$$

This choice yields

$$d_{\widetilde{\lambda}}(F) = 2^{-|L|} d_\lambda(F). \tag{4.4}$$

So to compute the wavelet leaders of F it suffices to look to those associated to λ in $\Lambda_j(\mathcal{C}_0)$.

For $\lambda \in \Lambda_j$ write

$$\frac{k}{2^j} = \frac{K}{2^J}, \quad \text{where} K \in \mathbb{Z}^m - (2\mathbb{Z})^m \text{ and } J \leq j. \tag{4.5}$$

Remark 4. Note that λ and $\widetilde{\lambda}$ share the same J.

- If $q \leq t$, let $a = \dfrac{1}{t} + \dfrac{2}{q}$.

  If $j \geq 1$ and $\lambda \in \Lambda_j(\mathcal{C}_0)$ put

  $$C_\lambda(F) = \frac{1}{j^a} 2^{-\frac{mJ}{t}}. \tag{4.6}$$

  Clearly $d_\lambda(F) = C_\lambda(F) = \frac{1}{j^a} 2^{-\frac{mJ}{t}}$. It is easy to show that $F \in B_t^{\frac{m}{t},q}$.
- Let now $t < q$. If r is a positive integer, then $\Lambda_r(\mathcal{C}_0)$ contains 2^{mr} dyadic cubes of side 2^{-r}. Let σ_r be a bijection between $\{0, \ldots, 2^r - 1\}^m$ and $\{0, \ldots, 2^{mr} - 1\}$.

  Let $D = \bigcup_{r \geq 1} \{2^{mr}, \ldots, 2^{mr+1} - 1\}$. For each $j \in D$, there exists a unique $r \in \mathbb{N}$ such that $2^{mr} \leq j \leq 2^{mr+1} - 1$. Let $m_j = 2^{j-r} \sigma_r^{-1}(j - 2^{mr})$ and λ_j be the associated cube in $\Lambda_j(\mathcal{C}_0)$. Put

  $$\forall\, j \in D \; C_{\lambda_j}(F) = \frac{1}{(j \ln(j)^2)^{\frac{1}{q}}}, \quad \text{and} \quad C_\lambda(F) = 0 \text{ else}. \tag{4.7}$$

For all $L \in \mathbb{Z}^m$ and all $j \geq 2$, the function F has at most only one non-vanishing wavelet coefficient in $\Lambda_j(\mathcal{C}_L)$. Thus $F \in B_t^{\frac{m}{t},q}$.

At scale j, denote by $R_j(\mathcal{C}_0)$ the set of all cubes $\lambda \in \Lambda_j(\mathcal{C}_0)$ such that $j < 2^{mJ}$ (where J was given in (4.5)). We have the following result.

Proposition 4.1. *There exists* $C > 0$ *such that*

$$\textit{for all } \lambda \in R_j(\mathcal{C}_0) \textit{there exists} \lambda' \subset \lambda \textit{such that } C_{\lambda'} \geq \frac{C}{J^{\frac{2}{q}} 2^{\frac{m}{q} J}}. \tag{4.8}$$

Proof. Let $\lambda \in R_j(\mathcal{C}_0)$. Let j' be such that $2^{mJ} \leq j' \leq 2^{mJ+1} - 1$ and $\lambda_{j'} \subset \lambda$. This is possible since when j' increases from 2^{mJ} to $2^{mJ+1} - 1$, $\frac{m_{j'}}{2^{j'}}$ takes all dyadic values $\frac{k}{2^J}$, where $k \in \left\{0, \ldots 2^J - 1\right\}^m$.

Thus for $\lambda' = \lambda_{j'}$

$$C_{\lambda'}(F) = \frac{1}{(j' \ln(j')^2)^{\frac{1}{q}}}.$$

This achieves the proof. □

If

$$R_j(\mathcal{C}_L) = \{\lambda \in \Lambda_j(\mathcal{C}_L) : j < 2^{mJ}\} \tag{4.9}$$

then (4.3) implies that there exists $C > 0$ such that

$$\forall\, L\, \forall\, \lambda \in R_j(\mathcal{C}_L) \quad \exists\, \lambda' \subset \lambda \;:\; C_{\lambda'} \geq \frac{C 2^{-|L|}}{J^{\frac{2}{q}} 2^{\frac{m}{q} J}}. \tag{4.10}$$

Actually, we can show the following result.

Proposition 4.2.

$$\forall\, L \in \mathbb{Z}^m \;\forall\, j \geq 2 \;\forall\, \lambda \in \Lambda_j(\mathcal{C}_L) \quad d_\lambda(F) \approx \begin{cases} \frac{2^{-|L|}}{(j\ln(j)^2)^{\frac{1}{q}}} & \text{if } 2^{mJ} < j \\ \frac{2^{-|L|}}{J^{\frac{2}{q}} 2^{\frac{m}{q}J}} & \text{else,} \end{cases}$$

where the notation $u \approx v$ means that there exist $C > 0$ independent of λ and j such that $\frac{v}{C} \leq u \leq Cv$.

Proof. Thanks to (4.4), we can assume that $j \geq 2$ and $\lambda \in \Lambda_j(\mathcal{C}_0)$. Let J defined by (4.5).

- If $r' < J$, then $C_{\lambda'}(F) = 0$, since at scale j' the only non-vanishing coefficient is located at $\frac{m_{j'}}{2^{j'}} = \frac{k'}{2^{r'}}$, for some $k' \in \{0, \dots 2^{r'} - 1\}^m$, then $J' \leq r' < J$ and the corresponding cube cannot be included in λ.
- If $r' \geq J$ then $j \leq 2^{mr'+1}$. Let r be the unique integer such that $2^{mr} \leq j < 2^{m(r+1)}$. We have necessarily $r \leq r'$.

  $*$ If $2^{mJ} < j$ then $J \leq r$. For any $j' \in D$ such that $2^{mr} \leq j' < 2^{mr+1}$ and $\lambda' \subset \lambda$, if $C_{\lambda'}(F) \neq 0$, then $C_{\lambda'}(F) = \frac{1}{(j'\ln(j')^2)^{\frac{1}{q}}} \approx \frac{1}{(j\ln(j)^2)^{\frac{1}{q}}}$. Thus

$$d_\lambda(F) = C_{\lambda'}(F) \approx \frac{1}{(j\ln(j)^2)^{\frac{1}{q}}}. \tag{4.11}$$

  Suppose that for all j' such that $2^{mr} \leq j' < 2^{mr+1}$ and $\lambda' \subset \lambda$ we have $C_{\lambda'}(F) = 0$. Since $J < r+1 < j$, then there exist only one j' such that $2^{m(r+1)} \leq j' < 2^{m(r+1)+1}$ and $\lambda_{j'} \subset \lambda$ (it suffices to take j' such that $\frac{m_{j'}}{2^{j'}} = \frac{k'}{2^{r+1}} = \frac{k}{2^j}$) and in this case we have

$$d_\lambda(F) = C_{\lambda'}(F) = \frac{1}{(j'\ln(j')^2)^{\frac{1}{q}}} \approx \frac{1}{(j\ln(j)^2)^{\frac{1}{q}}}. \tag{4.12}$$

  $*$ If $j \leq 2^{mJ}$, then since we should have $r' \geq J$, the best r' is J. Take j' such that $\frac{m_{j'}}{2^{j'}} = \frac{K}{2^J} = \frac{k}{2^j}$. Then $2^{mJ} \leq j' < 2^{mJ+1}$, $\lambda_{j'} \subset \lambda$ and

$$d_\lambda(F) = C_{\lambda'}(F) = \frac{1}{(j'\ln(j')^2)^{\frac{1}{q}}} \approx \frac{1}{J^{\frac{2}{q}} 2^{\frac{m}{q}J}}. \tag{4.13}$$

□

Let $q_1, q_2, t_1, t_2 > 0$. Let $i \in \{1, 2\}$. Put $b_i = \max\{q_i, t_i\}$ and let F_i be the function given by (4.2), (4.3) and(4.6) (resp. and (4.7)) if $q_i \leq t_i$ (resp. if $t_i < q_i$), where q, t are replaced by q_i and t_i.

Using the above results, clearly $\omega_{(F_1,F_2)}(p_1, p_2) = \omega^{\mathcal{C}_L}_{(F_1,F_2)}(p_1, p_2)$ for all $L \in \mathbb{Z}^m$, and we can directly show that (F_1, F_2) satisfies (2.21). But, since (F_1, F_2) will be in the G_δ-set, we will prove (2.21) on the entire G_δ-set in the next section.

4.2. The G_δ-set

Let (F_1, F_2) as above. If parameters t_i, q_i are finite then the product space $B_{t_1}^{\frac{m}{t_1},q_1} \times B_{t_2}^{\frac{m}{t_2},q_2}$ is separable (in the case where one or more of these parameters equals infinity, the reader can accommodate the idea of the construction done in [14] for the steps below). Let $(f_{1,n}, f_{2,n})_n$ be a dense sequence in $B_{t_1}^{\frac{m}{t_1},q_1} \times B_{t_2}^{\frac{m}{t_2},q_2}$. For a nonnegative integer n, define $g_{i,n}$ of the form (2.3) with

$$c_k(g_{i,n}) = c_k(f_{i,n}) \tag{4.14}$$

and

$$C_\lambda(g_{i,n}) = C_\lambda(F_i) \text{ if } j \geq n \text{ and } \lambda \in \Lambda_j \text{ and } C_\lambda(g_{i,n}) = C_\lambda(f_{i,n}) \text{ else.} \tag{4.15}$$

Clearly, the sequence $(g_{1,n}, g_{2,n})_n$ is dense in $B_{t_1}^{\frac{m}{t_1},q_1} \times B_{t_2}^{\frac{m}{t_2},q_2}$.
Let $a_i = \dfrac{1}{t_i} + \dfrac{2}{q_i}$ if $q_i \leq t_i$ and $a_i = \dfrac{2}{q_i}$ if $q_i > t_i$. If $t_i < q_i$, let C_i be a constant given by (4.8), (4.11), (4.12) and (4.13).

Let $C_i' = 1$ if $q_i \leq t_i$ and $C_i' = C_i$ if $t_i < q_i$.

Put

$$r_i(n) = \frac{C_i'}{2n^{a_i}} 2^{-mn/b_i}.$$

The residual set of $B_{t_1}^{\frac{m}{t_1},q_1} \times B_{t_2}^{\frac{m}{t_2},q_2}$ is

$$A = \bigcap_{L\in\mathbb{Z}^m} \bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} B(g_{1,n}, 2^{-|L|} r_1(n)) \times B(g_{2,n}, 2^{-|L|} r_2(n)), \tag{4.16}$$

where $B(g_{i,n}, 2^{-|L|} r_i(n))$ denotes the open ball in $B_{t_i}^{\frac{m}{t_i},q_i}$ of center $g_{i,n}$ and radius $2^{-|L|} r_i(n)$.

We have the following proposition.

Proposition 4.3. *If $(f_1, f_2) \in A$ and $L \in \mathbb{Z}^m$ then for infinitely many ns*

$$(f_1, f_2) \in B(g_{1,n}, 2^{-|L|} r_1(n)) \times B(g_{2,n}, 2^{-|L|} r_2(n)) \tag{4.17}$$

and

$$\forall\, \lambda \in \Lambda_n(\mathcal{C}_L)\ \forall\, i \in \{1, 2\} \quad d_\lambda(f_i) \geq \frac{1}{2}\, d_\lambda(F_i).$$

Proof. Clearly, if $(f_1, f_2) \in A$ and $L \in \mathbb{Z}^m$, then for infinitely many ns (4.17) holds. It follows that

$$\forall\, \lambda \in \Lambda_n(\mathcal{C}_L)\ \forall\, i \in \{1, 2\} \quad |C_\lambda(f_i) - C_\lambda(g_{i,n})| < 2^{-|L|} r_i(n). \tag{4.18}$$

If $q_i \leq t_i$, then thanks to the choice of $r_{i,n}$, we have for all $\lambda \in \Lambda_n(\mathcal{C}_L)$

$$d_\lambda(f_i) \geq |C_\lambda(f_i)| \geq C_\lambda(F_i) - 2^{-|L|} r_{i,n} \geq \frac{1}{2} C_\lambda(F_i) = \frac{1}{2} d_\lambda(F_i).$$

If $t_i < q_i$, we have seen that (4.11), (4.12) and (4.13) hold. In each case $L = 0$ and

$$d_\lambda(f_i) \geq |C_{\lambda'}(f_i)| \geq C_{\lambda'}(F_i) - 2^{-|L|} r_{i,n} \geq \frac{1}{2} C_{\lambda'}(F_i) = \frac{1}{2} d_\lambda(F_i).$$

Using (4.3), the last result remains valid for any $L \in \mathbb{Z}^m$. $\square$

Now we can achieve the proof of Theorem 2.2. Let A be the residual set (4.16). If $(f_1, f_2) \in A$ and $L \in \mathbb{Z}^m$, then for infinitely many ns (4.17) holds. Let δ_n be the integer part of $\dfrac{\log n}{m \log 2}$. Propositions 4.2 and 4.3 together with (4.9) imply that

$$\begin{aligned}
\sum_{\lambda \in \Lambda_n(\mathcal{C}_L)} (d_\lambda(f_1))^{p_1} (d_\lambda(f_2))^{p_2} &\geq \sum_{\lambda \in R_n} (d_\lambda(f_1))^{p_1} (d_\lambda(f_2))^{p_2} \\
&\geq C'' 2^{-|L|(p_1+p_2)} \sum_{J=\delta_n+1}^{n} \frac{2^{mJ}}{J^{a_1 p_1 + a_2 p_2}} 2^{-m(\frac{p_1}{b_1} + \frac{p_2}{b_2})J} \\
&\geq C'' \frac{2^{-|L|(p_1+p_2)}}{n^{a_1 p_1 + a_2 p_2}} \sum_{J=\delta_n+1}^{n} 2^{m(1 - \frac{p_1}{b_1} - \frac{p_2}{b_2})J} := H_n.
\end{aligned}$$

Thus

$$\begin{aligned}
\omega^{\mathcal{C}_L}_{(f_1,f_2)}(p_1, p_2) &\leq m + \liminf_{n \to +\infty} \frac{\log \left(\sum_{\lambda \in \Lambda_n(\mathcal{C}_L)} (d_\lambda(f_1))^{p_1} (d_\lambda(f_2))^{p_2} \right)}{\log 2^{-n}} \\
&\leq m + \liminf_{n \to +\infty} \frac{\log \left(\sum_{\lambda \in R_n(\mathcal{C}_L)} (d_\lambda(f_1))^{p_1} (d_\lambda(f_2))^{p_2} \right)}{\log 2^{-n}} \\
&\leq m + \liminf_{n \to +\infty} \frac{\log H_n}{\log 2^{-n}}.
\end{aligned}$$

- If $\frac{p_1}{b_1} + \frac{p_2}{b_2} < 1$, then $H_n \approx C'' \frac{2^{-|L|(p_1+p_2)}}{n^{a_1 p_1 a_2 p_2}} 2^{m(1 - \frac{p_1}{b_1} - \frac{p_2}{b_2})n}$. Thus

$$m + \liminf_{n \to +\infty} \frac{\log H_n}{\log 2^{-n}} = m(\frac{p_1}{b_1} + \frac{p_2}{b_2}).$$

  Thus

$$\omega^{\mathcal{C}_L}_{(f_1,f_2)}(p_1, p_2) \leq m(\frac{p_1}{b_1} + \frac{p_2}{b_2}).$$

- If $\frac{p_1}{b_1} + \frac{p_2}{b_2} \geq 1$, then $H_n \approx C'' \frac{2^{-|L|(p_1+p_2)}}{n^{a_1 p_1 a_2 p_2}} 2^{m(1 - \frac{p_1}{b_1} - \frac{p_2}{b_2})\delta_n} \approx n^\alpha$, for a constant α that we do not need to precise. Thus

$$m + \liminf_{n \to +\infty} \frac{\log H_n}{\log 2^{-n}} = m.$$

  Thus

$$\omega^{\mathcal{C}_L}_{(f_1,f_2)}(p_1, p_2) \leq m.$$

Gathering these upper bounds with the lower bounds already obtained in Theorem 2.1, we get (2.21). Result (2.22) is a consequence of Lemma 3.3. $\square$

5. Proof of Theorem 2.3

1. In [27], the following result is proved.

Proposition 5.1. *Let $q \leq 1$ and $t > 0$. Put $b = \max\{q, t\}$. If $f \in B_t^{\frac{m}{t},q}$ then*
$$\forall\ h \geq 0 \quad dim\, E_f^h \leq \min\{m, bh\}.$$

By (1.11), if $(f_1, f_2) \in B_{t_1}^{\frac{m}{t_1},q_1} \times B_{t_2}^{\frac{m}{t_2},q_2}$ then
$$dim\, E_{f_1}^{h_1} \cap E_{f_2}^{h_2} \leq \min\left\{dim\, E_{f_1}^{h_1}, dim\, E_{f_2}^{h_2}\right\}.$$
From Proposition 5.1 and the upper bound (1.11)
$$\forall\ h_1 \geq 0\ \forall\ h_2 \geq 0 \quad \dim E_{f_1}^{h_1} \cap E_{f_2}^{h_2} \leq \min\{m, b_1h_1, b_2h_2\}. \tag{5.1}$$
We will now show that this upper bound is optimal in the Baire sense.

Let $\alpha \geq 1$. Let $L \in \mathbb{Z}^m$. For each scale j, denote by $\Lambda_j^L(\alpha)$ the dyadic cubes $\lambda \in \Lambda_j(\mathcal{C}_L)$ such that $J = [\frac{j}{\alpha}]$.

Let A be the residual set (4.16) of the space $B_{t_1}^{\frac{m}{t_1},q_1} \times B_{t_2}^{\frac{m}{t_2},q_2}$. Let $(f_1, f_2) \in A$. For each $L \in \mathbb{Z}^m$, fix the sequence of infinitely many ns such that (4.17) holds. Let $K^L(\alpha)$ be the set of points x that belong to $\Lambda_n^L(\alpha)$ for the above ns. Using [16,23,24], we have the following result.

Proposition 5.2. *If $\alpha \geq 1$ then $dim\, K^L(\alpha) = \frac{m}{\alpha}$ and there exists a σ-finite measure μ_α^L carried by $K^L(\alpha)$ such that, if $E \subset K^L(\alpha)$ and $dim\, E < \frac{m}{\alpha}$ then $\mu_\alpha^L(E) = 0$.*

By applying (2.12) and Proposition 4.3, we have
$$\forall\ i \in \{1, 2\}\ \forall\ x \in K^L(\alpha) \quad h_{f_i}(x) \leq \frac{m}{\alpha b_i}. \tag{5.2}$$
Let $h_1, h_2 \geq 0$. Put $\beta = \min\{m, b_1h_1, b_2h_2\}$. Put $\alpha = \frac{m}{\beta}$. Result (5.2) implies that
$$K^L(\alpha) \subset E_{f_1}^{h_1} \cap E_{f_2}^{h_2} \cap \mathcal{C}_L. \tag{5.3}$$
From Proposition 5.2, it follows that
$$\min\{m, b_1h_1, b_2h_2\} = \frac{m}{\alpha} \leq \dim E_{f_1}^{h_1} \cap E_{f_2}^{h_2} \cap \mathcal{C}_L. \tag{5.4}$$
Thus using (2.22) and (5.1), we deduce that (2.23) holds.

2. Let $(f_1, f_2) \in A$. For $h_1 \in [0, \frac{m}{b_1}]$ and $h_2 \in [0, \frac{m}{b_2}]$, we have
$$\dim E_{f_1}(h_1) \cap E_{f_2}(h_2) \leq \dim E_{f_1}^{h_1} \cap E_{f_2}^{h_2} = \min\{b_1h_1, b_2h_2\}.$$
If moreover $h_1b_1 = h_2b_2$, then
$$\dim(E_{f_1}(h_1) \cap E_{f_2}(h_2)) \leq \dim(E_{f_1}^{h_1} \cap E_{f_2}^{h_2}) = b_1h_1 = b_2h_2.$$
Let us show the lower bound. Take $\alpha = b_1h_1$. Clearly
$$K_L(\alpha) \subset E_{f_1}^{h_1} \cap E_{f_2}^{h_2} \cap \mathcal{C}_L.$$
Split $E_{f_1}^{h_1} \cap E_{f_2}^{h_2} \cap \mathcal{C}_L$ as
$$E_{f_1}^{h_1} \cap E_{f_2}^{h_2} \cap \mathcal{C}_L = (E_{f_1}(h_1) \cap E_{f_2}(h_2) \cap \mathcal{C}_L) \cup S_L(h_1, h_2) \cup T_L(h_1, h_2),$$

where

$$S_L(h_1,h_2)=\bigcup_{N\geq 1} E_{f_1}^{h_1-\frac{1}{N}}\cap E_{f_2}^{h_2}\cap \mathcal{C}_L$$

and

$$T_L(h_1,h_2)=\bigcup_{N\geq 1} E_{f_1}^{h_1}\cap E_{f_2}^{h_2-\frac{1}{N}}\cap \mathcal{C}_L.$$

Let μ_α^L be the measure considered in Proposition 5.2. Since for all $N\geq 1$

$$\begin{aligned}\dim K_L(\alpha)\cap E_{f_1}^{h_1-\frac{1}{N}}\cap E_{f_2}^{h_2} &\leq \dim E_{f_1}^{h_1-\frac{1}{N}}\cap E_{f_2}^{h_2}\cap \mathcal{C}_L\\ &=(h_1-\frac{1}{N})b_1<\frac{m}{\alpha},\end{aligned}$$

then Proposition 5.2 yields

$$\mu_\alpha^L(K_L(\alpha)\cap E_{f_1}^{h_1-\frac{1}{N}}\cap E_{f_2}^{h_2})=0.$$

It follows that

$$\mu_\alpha^L(S_L(h_1,h_2))=\sup_{N\geq 1}\mu_\alpha^L(K_L(\alpha)\cap E_{f_1}^{h_1-\frac{1}{N}}\cap E_{f_2}^{h_2})=0.$$

Similar argument yields

$$\mu_\alpha^L(T_L(h_1,h_2))=0.$$

Therefore,

$$0<\mu_\alpha^L(K(\alpha))=\mu_\alpha^L(K_L(\alpha)\cap E_{f_1}(h_1)\cap E_{f_2}(h_2)).$$

By Proposition 5.2, we deduce that

$$\dim K_L(\alpha)\cap E_{f_1}(h_1)\cap E_{f_2}(h_2)=\frac{m}{\alpha}.$$

Consequently,

$$h_1b_1=h_2b_2=\frac{m}{\alpha}\leq \dim E_{f_1}(h_1)\cap E_{f_2}(h_2)\cap \mathcal{C}_L.$$

Finally,

$$\begin{aligned}\dim E_{f_1}(h_1)\cap E_{f_2}(h_2)\cap \mathcal{C}_L &= \dim\left(E_{f_1}(h_1)\cap E_{f_2}(h_2)\right)=h_1\\ b_1 &= h_2b_2=m+\omega^*_{(f_1,f_2)}(h_1,h_2).\end{aligned}$$

□

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