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MULTIFRACTAL FORMALISM OF OSCILLATING SINGULARITIES FOR RANDOM WAVELET SERIES

MOURAD BEN SLIMANE* and BORHEN HALOUANI†
Department of Mathematics, College of Science
King Saud University, P. O. Box 2455
Riyadh 11451, Saudi Arabia
**mbenslimane@ksu.edu.sa*
†halouani@ksu.edu.sa

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Abstract

The oscillating multifractal formalism is a formula conjectured by Jaffard expected to yield the spectrum $d(h, \beta)$ of oscillating singularity exponents from a scaling function $\zeta(p, s')$, for $p > 0$ and $s' \in \mathbb{R}$, based on wavelet leaders of fractional primitives $f_{-s'}$ of f. In this paper, using some results from Jaffard *et al.*, we first show that $\zeta(p, s')$ can be extended on $p \in \mathbb{R}$ to a function that is concave with respect to $p \in \mathbb{R}$ and independent on orthonormal wavelet bases in the Schwartz class. We also establish its concavity with respect to s' when $p > 0$. Then, we prove that, under some assumptions, the extended scaling function $\zeta(p, s')$ is the Legendre transform of the wavelet leaders density of $f_{-s'}$. Finally, as an application, we study the validity of the extended oscillating multifractal formalism for random wavelet series (under the assumption of independence and laws depending only on the scale).

Keywords: Oscillating Singularity Exponents; Multifractal Formalism; Scaling Function; Random Wavelet Series; Oscillation Spaces; Wavelet Leaders; Fractional Primitives; Wavelet Leaders Density; Wavelet Leaders Profile.

1. INTRODUCTION

Let $f : \mathbb{R} \to \mathbb{R}$ be a locally bounded function. Recall that $f \in C^h(x_0)$ for $h > 0$ if there exist a polynomial P of degree smaller than h and a constant C such that

$$\forall\, x \in \mathbb{R} \quad |f(x) - P(x - x_0)| \leq C|x - x_0|^h. \qquad (1)$$

The Hölder exponent $h(x_0)$ (also denoted $h_f(x_0)$) of f at x_0 is the sup of all values of h such that $f \in C^h(x_0)$.

The Hölder exponent, as a measure of pointwise regularity of functions, is a powerful tool in many applications such as image analysis or signal analysis.[1,2]

The Hölder spectrum of f is the Hausdorff dimension $d(h) = d_f(h)$ of the set of points x where $h(x) = h$ (we take $\dim(\emptyset) = -\infty$).

We also say that $f \in C^h(\mathbb{R})$ for $h > 0$ if (1) holds for any x_0 in $\mathbb{R}$ with a uniform C.

The (standard) multifractal formalism was introduced by Frisch and Parisi[3] in the context of fully developed turbulence, and alternatively by Arneodo *et al.*[4] and Jaffard.[5] It proposes to compute the Hölder spectrum using the formula

$$d(h) = \inf_{p \geq p_c} (ph - \eta(p) + 1), \qquad (2)$$

where

$$\eta(p) = \sup\{s : f \in B_p^{s/p,\infty}(\mathbb{R})\}, \qquad (3)$$

where $B_p^{s/p,\infty}(\mathbb{R})$ denotes the Besov spaces and p_c is critical value for which $\eta(p_c) = 1$.

The validity of (2) has been the subject of many papers. We would not give a detailed review, let us just mention that this validity has been proved under self-similarity assumptions on f,[5–8] or for a class of particular random processes,[9] or even for specific functions f,[10,11] or generically in the sense of Baire and prevalence.[12,13]

However, the validity never holds in complete generality: in Refs. 14 and 15, it has been proved that the (standard) multifractal formalism fails in the case where the function involves very oscillating behaviors; the Hölder exponent does not take into account the local oscillations of the function. Indeed a given Hölder exponent h at x_0 allows for different behaviors near x_0: for instance cusp-like singularities, such as $|x - x_0|^h$ or very oscillatory behaviors, such as

$$f_{h,\beta}(x) = |x - x_0|^h \sin\left(\frac{1}{|x - x_0|^\beta}\right), \qquad (4)$$

for $\beta > 0$. The functions $f_{h,\beta}$ are the most simple examples of chirps at x_0. In signal analysis, this notion is expected to give a model for functions whose "instantaneous frequency" increases fast at some time (see Ref. 16).

In Ref. 17, Mélot showed that formula (2) is only adapted to cusp singularities (i.e. singularities with oscillation exponent $\beta = 0$). Since

$$\int_0^x t^h \sin\left(\frac{1}{t^\beta}\right) dt = \frac{x^{h+\beta+1}}{\beta} \cos\left(\frac{1}{x^\beta}\right) - \frac{h+\beta+1}{\beta} x^{h+\beta+1} \times \int_0^1 s^{h+\beta} \cos\left(\frac{1}{(sx)^\beta}\right) ds, \qquad (5)$$

then contrary to functions with cusp singularities, the primitive of the oscillating function (4) has a Hölder exponent $h + 1 + \beta$ at x_0 which is different from $h+1$. This remark motivated the following definitions introduced by Jaffard and Meyer in Ref. 16.

Definition 1. Let $h \geq 0$ and $\beta > 0$. A function $f \in L^\infty(\mathbb{R})$ is a (h, β)-type chirp at x_0 if for each $n \in \mathbb{N}$, f can be written as $f = f_n^{(n)}$ (nth derivative) with $f_n \in C^{h+n(\beta+1)}(x_0)$.

In this case, f can be written also as

$$f(x) = |x - x_0|^h g_{\pm}\left(\pm \frac{1}{|x - x_0|^\beta}\right) + R(x - x_0), \qquad (6)$$

where $\pm$ stands for the sign of $x - x_0$, the two functions $g_+(t)$ and $g_-(t)$ are defined on $[T, \infty)$ (for some $T > 0$) and are indefinitely oscillating, i.e. all their primitives are bounded, and $R(x)$ is C^∞ in a neighborhood of the origin.

One immediately meets some difficulties when using this definition for experimental data. Indeed it is not stable when one adds to f a function which is arbitrarily smooth, but not C^∞. Consider for example the function

$$f(x) = x \sin\left(\frac{1}{x^\beta}\right) + |x|^{3/2}, \qquad (7)$$

then for n large enough such that $1 + n(\beta + 1) > \frac{3}{2} + n$, the Hölder exponent $h_{f_n}(0)$ equals $\frac{3}{2} + n$. This drawback can be avoided by introducing a slightly different definition of oscillating singularities which agrees with the definition of a chirp for functions such as (6), and has the required stability properties with respect to the addition of smooth noise.

Consider

$$f_{h,\beta,\mathcal{O}}(x) = |x-x_0|^h g_{\pm}\left(\pm\frac{1}{|x-x_0|^\beta}\right) + \mathcal{O}(|x-x_0|^{h'}), \quad (8)$$

where $h' > h$; the first term describes the local behavior of $f_{h,\beta,\mathcal{O}}$ near x_0, so that, if $x \mapsto \int_0^x g(t) \times dt \in L^\infty(\mathbb{R})$, the oscillating singularity exponent at x_0 should be (h, β).

Definition 2. Let $f \in L^2(\mathbb{R})$. The fractional primitive of order t of f is the function $(Id - \frac{d^2}{dx^2})^{-t/2}(f)$, where the operator $(Id - \frac{d^2}{dx^2})^{-t/2}$ is the convolution operator which amounts multiplying the Fourier transform of f with $(1+|\xi|^2)^{-t/2}$.

If f is locally bounded, the local fractional primitive of order t at x_0 of f is the function

$$f_t := \left(Id - \frac{d^2}{dx^2}\right)^{-t/2}(\phi f),$$

where ϕ is a C^∞ compactly supported function satisfying $\phi(x_0) = 1$.

The Hölder exponent of the function f_t is denoted $h_t(x_0)$.

Note that $h_t(x_0)$ does not depend on the function ϕ. In the case of the function $f_{h,\beta,\mathcal{O}}$ defined by (8), for t small enough, $h_t(x_0) = h + (1+\beta)t$: the increase of pointwise Hölder regularity at x_0 after a fractional integration of very small order t is $(1+\beta)t$. For the example (7), for $t > 0$ small enough such that $1 + t(\beta+1) < \frac{3}{2} + t$ we get $h_t(0) = 1 + t(\beta+1)$. This remark motivated the following definition of Ref. 14.

Definition 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a locally bounded function and x_0 such that $h_t(x_0) < \infty$. The oscillating singularity exponents of f at x_0 are defined by

$$\left(h(x_0), \beta(x_0) = \frac{\partial}{\partial t} h_t(x_0)|_{t=0} - 1\right).$$

This definition makes sense because, for a given x_0, the function $t \mapsto h_t(x_0)$ is concave (with slope ≥ 1) (see Ref. 14), hence it is differentiable on the right, with a possible infinite derivative so that $\beta(x_0)$ can be infinite.

If we want to study oscillating singularities located in a signal, we are naturally led to define the following spectrum.

Definition 4. The spectrum $d(h, \beta) = d_f(h, \beta)$ of oscillating singularities of a function f is the Hausdorff dimension of the set of points where f has oscillating singularity exponents (h, β).

In Ref. 18, using heuristic arguments, Jaffard derived an oscillating multifractal formalism which yields $d(h, \beta)$ from the knowledge of oscillation spaces $\mathcal{O}_p^{s,s'}(\mathbb{R})$, for $p > 0$ (see Definition 5), to which the function f belongs. That spaces quantify the degree of correlations between positions of large wavelet coefficients through the scales; for $p > 0$, if

$$\zeta(p, s') = \zeta_f(p, s') = \sup\{s; f \in \mathcal{O}_p^{s/p,s'}(\mathbb{R})\}, \quad (9)$$

then the heuristic formula is

$$\zeta(p, s') = \inf_h \inf_\beta (hp - s'(1+\beta)p - d(h, \beta)). \quad (10)$$

So, if $d(h, \beta)$ is concave then

$$d(h, \beta) = \inf_p \inf_{s'} (hp - s'(1+\beta)p - \zeta(p, s')). \quad (11)$$

Jaffard[18] proved that this formula allows to recover the increasing part of the spectrum in the case of lacunary wavelet series. Obtaining the decreasing part of the spectrum would correspond to an infimum in (11) obtained for negative values of p.

Lacunary wavelet series is a simplified case of random wavelet series: the nonzero coefficients at a given scale take only one value. Random wavelet series (see Ref. 19) are obtained by first choosing an (almost) arbitrary sequence of histograms of wavelet coefficients at each scale, and then drawing at random each wavelet coefficients at each scale inside the corresponding histogram, independently. Random wavelet series are compatible with several turbulence models that have been proposed in the past (see Ref. 20). Other definitions of random wavelet series have been the subject of many papers.[21–24]

In Sec. 2, we first recall the definition of oscillation spaces, then we extend $\zeta(p, s')$ to $p \in \mathbb{R}$. We show that it is concave with respect to $p \in \mathbb{R}$ and independent on orthonormal wavelet bases in the Schwartz class (using some results from Jaffard *et al.*[25]) and we prove its concavity with respect to s' when $p > 0$, see Proposition 1. In Sec. 3, we obtain a relationship for the scaling function $\omega(p)$ (respectively, $\zeta(p, s')$) for $p \in \mathbb{R}$ based on the large deviation quantities derived from the distributions of wavelet leaders, see Proposition 2 (respectively, see Corollary 1). We thus extend to $p \in \mathbb{R}$ an earlier result obtained by Jaffard[26] for $p > 0$. In Sec. 4, we recall some definitions, notations and results (due to Aubry and Jaffard[19]) concerning the random wavelet series. As an application, in Sec. 5, we use the generalized scaling function $\zeta(p, s')$ to study the validity of both the multifractal formalism of oscillating singularities and its inverse formula (11) and (10) for random wavelet series, see Theorem 3. In Sec. 6, we give a short conclusion.

2. SCALING FUNCTION

We consider functions on the unit torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (1-periodic functions). Extensions to $\mathbb{R}$ and higher dimension are straightforward. Let ψ be a mother wavelet such that the constant function 1, together with the periodized functions

$$2^{j/2}\psi_{j,k}(x) := 2^{j/2}\sum_{l\in\mathbb{Z}}\psi(2^j(x-l)-k),$$

$j \geq 0$, $k \in \{0,\ldots,2^j-1\}$, form an orthonormal basis of the space $L^2(\mathbb{T})$ (see Refs. 27 and 28). Since the L^∞ normalization for wavelets is more convenient for our purpose, we denote

$$C_{j,k} = 2^j\int_{[0,1]} f(t)\psi_{j,k}(t)dt, \tag{12}$$

the wavelet coefficients of a function f in $L^2(\mathbb{T})$ (with the usual modification when f is a tempered distribution periodic over $\mathbb{Z}$).

We will use the following simpler notations; λ and λ' will denote, respectively, the intervals $\lambda_{j,k} = [k2^{-j}, k2^{-j} + 2^{-j})$ and $\lambda_{j',k'} = [k'2^{-j'}, k'2^{-j'} + 2^{-j'})$, C_λ will denote the coefficient $C_{j,k}$, and ψ_λ will denote the wavelet $\psi_{j,k}$. If j is fixed, we will denote by Λ_j the set of all intervals $\lambda = \lambda_{j,k}$ where $k \in \{0,\ldots,2^j-1\}$.

Let us first recall the definition of oscillation spaces $\mathcal{O}_p^{s,s'}(\mathbb{T})$ given by Jaffard in Ref. 18.

Definition 5. Let $p > 0$, and s, $s' \in \mathbb{R}$; a function $f \in L^\infty(\mathbb{T})$ belongs to the oscillation space $\mathcal{O}_p^{s,s'}(\mathbb{T})$ if its wavelet coefficients (12) satisfy

$$\begin{aligned}&\exists\, C > 0 \quad \forall\, j \geq 0\\ &S(p,j,s') = S_f(p,j,s')\\ &\qquad = \sum_{\lambda\in\Lambda_j}\left(\sup_{\lambda'\subset\lambda}|C_{\lambda'}2^{s'j'}|\right)^p\\ &\qquad \leq C2^{-spj}.\end{aligned} \tag{13}$$

In Ref. 26, Jaffard proved that, for either $s \geq 0$ or $s \leq -1/p$, oscillation spaces are a variation on the definition of Besov or Sobolev spaces. Recall that Besov spaces $B_p^{s,q} = B_p^{s,q}(\mathbb{T})$ for $(s \in \mathbb{R}, 0 < p \leq \infty, 0 < q \leq \infty)$ are expressed by simple conditions (mixed $\ell^p - \ell^q$ norms) on wavelet coefficients (see Ref. 29):

$$f \in B_p^{s,q} \Leftrightarrow \left(\sum_j\left(\sum_{\lambda\in\Lambda_j}|C_\lambda 2^{(s-\frac{1}{p})j}|^p\right)^{q/p}\right)^{1/q} < \infty, \tag{14}$$

(with the usual modification when $p=\infty$ and/or $q=\infty$). On the contrary the spaces $\mathcal{O}_p^{s,s'}(\mathbb{R})$ for $-1/p < s < 0$ cannot be sharply imbedded between Sobolev spaces, and thus are new spaces of really different nature.

Remark 1. If either $f \in L^\infty(\mathbb{T})$ or $f \in B_\infty^{0,\infty}(\mathbb{T})$, then there exists $C > 0$ such that

$$\forall\,\lambda\in\Lambda \quad |C_\lambda| \leq C. \tag{15}$$

If $s' \leq 0$ then quantities $\sup_{\lambda'\subset\lambda}|C_{\lambda'}2^{s'j'}|$ are finite.

The scaling function $\zeta(p,s')$ given in (9) satisfies

$$\zeta(p,s') = \liminf_{j\to\infty}\frac{\log(S(p,j,s'))}{\log(2^{-j})}. \tag{16}$$

Note that (16) is a theorem if $p > 0$ and a definition if $p \leq 0$.

Remark 2. Quantities $C_{\lambda'}2^{s'j'}$ are the wavelet coefficients, at scale j', of a pseudo-fractional primitive $\tilde{f}_{-s'}$, of f, of order $-s'$. Therefore

$$\zeta_f(p,s') = \zeta_{\tilde{f}_{-s'}}(p,0). \tag{17}$$

Quantities

$$d_\lambda := \sup_{\lambda'\subset\lambda}|C_{\lambda'}| \tag{18}$$

are called the wavelet leaders of f. Therefore, quantities

$$d_{\lambda,t} := \sup_{\lambda'\subset\lambda}|C_{\lambda'}2^{-tj'}| \tag{19}$$

are the wavelet leaders of $\tilde{f}_t$. In Refs. 14, 17 and 30, it is proved that $h_t(x_0)$ (given in Definition 2) amounts to compute the Hölder exponent at x_0 of $\tilde{f}_t$.

In Ref. 25, Jaffard *et al.* conjectured a multifractal formalism expected to yield the spectrum $d(h)$ of Hölder exponents from a scaling function defined for $p \in \mathbb{R}$ by

$$\omega(p) = \omega_f(p) = \liminf_{j\to\infty}\frac{\log(\sum_{\lambda\in\Lambda_j}(d_\lambda)^p)}{\log(2^{-j})}. \tag{20}$$

Clearly if $\zeta(p,0)$ is extended for $p \in \mathbb{R}$ as in (16) with $s' = 0$ then

$$\omega(p) = \zeta(p,0). \tag{21}$$

We will show the following properties for the scaling function $\zeta(p,s')$.

Proposition 1. *The function $p \mapsto \zeta(p,s')$ given in* (16), *extended on the entire space of real numbers, satisfies*

$$\zeta(p,s') = \omega_{\tilde{f}_{-s'}}(p) = \liminf_{j\to\infty} \frac{\log(\sum_{\lambda\in\Lambda_j}(\sup_{\lambda'\subset\lambda}|C_{\lambda'}2^{s'j'}|)^p)}{\log(2^{-j})}. \quad (22)$$

It is concave on $\mathbb{R}$ with respect to p and independent on orthonormal wavelet bases in the Schwartz class.

Moreover, if $p > 0$ then $s' \mapsto \zeta(p,s')$ is concave.

Proof of Proposition 1. In Ref. 25, Jaffard *et al.* proved that the scaling function $\omega(p)$ is concave on $\mathbb{R}$ (see also Ref. 31) and independent on orthonormal wavelet bases in the Schwartz class. Consequently, using (17) and (21), the function $p \mapsto \zeta_f(p,s')$ is concave and independent on orthonormal wavelet bases in the Schwartz class. It also satisfies (22).

Let us now show the second result; let $s'_2 = \alpha s'_1 + (1-\alpha)s'_3$ with $0 < \alpha < 1$. We have

$$|C_{\lambda'}2^{s'_2 j'}| = |C_{\lambda'}2^{s'_1 j'}|^{\alpha}|C_{\lambda'}2^{s'_3 j'}|^{(1-\alpha)}. \quad (23)$$

If $p > 0$, using the notations (19), it follows that

$$(d_{\lambda,-s'_2})^p \le (d_{\lambda,-s'_1})^{\alpha p}(d_{\lambda,-s'_3})^{(1-\alpha)p}.$$

By the Hölder inequality

$$\sum_{\lambda\in\Lambda_j}(d_{\lambda,-s'_2})^p \le \left(\sum_{\lambda\in\Lambda_j}(d_{\lambda,-s'_1})^p\right)^{\alpha} \times \left(\sum_{\lambda\in\Lambda_j}(d_{\lambda,-s'_3})^p\right)^{(1-\alpha)}.$$

By definition, $\forall\,\varepsilon > 0$, $\exists\, C > 0$ such that, for j large enough,

$$\sum_{\lambda\in\Lambda_j}(d_{\lambda,-s'_1})^p \le C2^{-j\zeta(p,s'_1)}2^{\varepsilon j}$$

and

$$\sum_{\lambda\in\Lambda_j}(d_{\lambda,-s'_3})^p \le C2^{-j\zeta(p,s'_3)}2^{\varepsilon j}.$$

Therefore,

$$\sum_{\lambda\in\Lambda_j}(d_{\lambda,-s'_2})^p \le C2^{-j\alpha\zeta(p,s'_1)}2^{-j(1-\alpha)\zeta(p,s'_3)}2^{\varepsilon j}.$$

Again, by definition, there exists a sequence $j_n \to \infty$ such that

$$\sum_{\lambda\in\Lambda_{j_n}}(d_{\lambda,-s'_1})^p \ge C2^{-j_n\zeta(p,s'_2)}2^{-\varepsilon j_n}.$$

It follows that

$$\zeta(p,s'_2) + \varepsilon \ge \alpha\zeta(p,s'_1) + (1-\alpha)\zeta(p,s'_3) - \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain the concavity of $s' \mapsto \zeta(p,s')$ for $p > 0$. □

3. RELATIONSHIPS BETWEEN SCALING FUNCTIONS AND DISTRIBUTIONS OF WAVELET LEADERS

Our purpose in this section is to give a relationship for the scaling function $\omega(p)$ based on the large deviation quantities derived from the distributions of wavelet leaders see Refs. 32 and 33 where this technique is discussed). Let us begin by some basic definitions concerning the distributions of wavelet leaders. Let f be a periodic tempered distribution. Let $\alpha \in \mathbb{R}$. For each j, let

$$M_j(\alpha) = \mathrm{Card}\{0 \le k < 2^j; d_\lambda \ge 2^{-\alpha j}\}, \quad (24)$$

where Card means cardinality.

The wavelet leaders density σ is defined by

$$\sigma(\alpha) = \sigma_f(\alpha) = \inf_{\varepsilon>0}\sigma(\alpha,\varepsilon), \quad (25)$$

where

$$\sigma(\alpha,\varepsilon) = \limsup_{j\mapsto\infty}\frac{\log(M_j(\alpha+\varepsilon) - M_j(\alpha-\varepsilon))}{\log(2^j)}. \quad (26)$$

A heuristic interpretation is that at scale j (when $j \to \infty$): there are about $2^{\sigma(\alpha)j}$ wavelet leaders of size of order $2^{-\alpha j}$.

The domain of definition of the function $\sigma(\alpha)$ is the set of α such that $\sigma(\alpha) \ne -\infty$, we denote it by $\mathrm{Dom}(\sigma)$.

Using some ideas from the proofs of Proposition 2.2 in Ref. 19, we will extend to $p \in \mathbb{R}$ an earlier result obtained by Jaffard (relation (38) in Ref. 26 page 49) which relates $\omega(p)$ for $p > 0$ to the distributions of wavelet leaders.

Proposition 2. *Let $\omega(p)$ and $\sigma(\alpha)$ be defined as in* (20) *and* (25). *If f is a periodic tempered distribution, then*

$$\forall\, p \in \mathbb{R} \quad \omega(p) = \inf_{\alpha\in\mathbb{R}}(\alpha p - \sigma(\alpha)). \quad (27)$$

Proof of Proposition 2. Since f is a periodic distribution, then f has finite order, and therefore f has a minimal (perhaps negative) uniform Hölder regularity. Therefore, there exist $\alpha_{\min} \in \mathbb{R}$ and $C > 0$ such that

$$\forall \lambda \quad |C_\lambda| \leq C 2^{-\alpha_{\min} j}.$$

It follows that

$$\mathrm{Dom}(\sigma) \subset [\alpha_{\min}, \infty). \tag{28}$$

- We first prove the upper bound in (27); fix $\alpha \geq \alpha_{\min}$ and $\delta > 0$. For any $\varepsilon > 0$ there exists a sequence (j_n) with $\lim_{n\to\infty} j_n = \infty$ such that

$$M_{j_n}(\alpha+\varepsilon) - M_{j_n}(\alpha-\varepsilon)) \geq 2^{j_n(\sigma(\alpha)-\delta)},$$

therefore

$$\forall p \in \mathbb{R} \quad \sum_{\lambda\in\Lambda_j} (d_\lambda)^p \geq 2^{j_n(\sigma(\alpha)-\delta)} 2^{-\alpha j_n p} 2^{-\varepsilon j_n |p|}.$$

Taking logarithms and letting j_n tend to ∞, we obtain

$$\forall p \in \mathbb{R} \quad \omega(p) \leq \alpha p + \varepsilon|p| - (\sigma(\alpha) - \delta)$$

for all $\delta > 0$ and $\varepsilon > 0$. Finally, letting δ and ε tend to 0 we see that

$$\forall p \in \mathbb{R} \quad \omega(p) \leq \alpha p - \sigma(\alpha).$$

Since $\alpha \geq \alpha_{\min}$ was arbitrary this inequality implies that

$$\forall p \in \mathbb{R} \quad \omega(p) \leq \inf_{\alpha\geq\alpha_{\min}} (\alpha p - \sigma(\alpha)).$$

- We now prove the lower bound in (27) for $p > 0$; let $A > 0$ be large enough and $\delta > 0$. For all $\alpha \in [\alpha_{\min}, A]$, there exists $\varepsilon > 0$ such that for all $\varepsilon' \leq \varepsilon$

$$|\sigma(\alpha, \varepsilon') - \sigma(\alpha)| \leq \delta.$$

Thus, there exist $\alpha_1, \ldots, \alpha_N$ and $\varepsilon_1 \leq \varepsilon, \ldots, \varepsilon_N \leq \varepsilon$ such that $[\alpha_{\min}, A]$ is covered by the intervals $[\alpha_i - \varepsilon_i, \alpha_i + \varepsilon_i]$, and for all $i \in \{1, \ldots, N\}$, $|\sigma(\alpha_i, \varepsilon_i) - \sigma(\alpha_i)| \leq 2\delta$. Thus, for each $(\alpha_i, \varepsilon_i)$ there exists J_i such that

$$\forall j \geq J_i, \quad M_j(\alpha_i+\varepsilon_i) - M_j(\alpha_i-\varepsilon_i)) \leq 2^{(\sigma(\alpha_i)+2\delta)j}.$$

Taking for J the maximum of the J_i, it follows that $\forall j \geq J$,

$$\forall p > 0$$

$$\sum_{\lambda\in\Lambda_j} (d_\lambda)^p \leq \sum_{i=1}^{N} (2^{j(\sigma(\alpha_i)+2\delta)} 2^{-\alpha j p} 2^{\varepsilon_i j p}) + 2^j 2^{-Ajp}$$

(the last term corresponds to wavelet leaders smaller than 2^{-Aj}). Thus,

$$\forall j \geq J, \quad \forall p > 0$$

$$\sum_{\lambda\in\Lambda_j} (d_\lambda)^p \leq N 2^{\sup_\alpha (j(\sigma(\alpha_i)-\alpha p))} 2^{(2\delta+\varepsilon p)j} + 2^j 2^{-Ajp}.$$

Taking logarithms and letting ε and δ (respectively A) tend to 0 and (respectively ∞), we see that

$$\forall p > 0 \quad \omega(p) \geq -\sup_{\alpha\geq\alpha_{\min}} (-\alpha p + \sigma(\alpha)) = \inf_{\alpha\geq\alpha_{\min}} (\alpha p - \sigma(\alpha)).$$

- Let us now prove the lower bound in (27) for $p < 0$. We first assume that $\mathrm{Dom}(\sigma)$ is bounded, it follows that $K := \overline{\mathrm{Dom}(\sigma)}$ is compact. By replacing the above interval $[\alpha_{\min}, A]$ by K, we get

$$\forall j \geq J \quad \forall p < 0$$

$$\sum_{\lambda\in\Lambda_j} (d_\lambda)^p \leq \sum_{i=1}^{N} 2^{j(\sigma(\alpha_i)+\delta)} 2^{-\alpha j p} 2^{-\varepsilon_i j p}.$$

Therefore, as previously

$$\forall p < 0 \quad \omega(p) \geq \inf_{\alpha\geq\alpha_{\min}} (\alpha p - \sigma(\alpha)).$$

We now consider the general case; let $A > 0$, denote by $\omega(A, p)$ the scaling function obtained with the sequence $\{d_\lambda : d_\lambda \geq 2^{-Aj}\}$. We just proved that

$$\forall p < 0 \quad \omega(A, p) \geq \inf_{\alpha_{\min}\leq\alpha\leq A} (\alpha p - \sigma(\alpha)). \tag{29}$$

Since $\omega(A, p)$ decreases when A increases then

$$\omega(p) = \lim_{A\to\infty} \omega(A, p) = \inf_{A>0} \omega(A, p).$$

Then, from (29) we deduce that

$$\forall p < 0 \quad \omega(p) \geq \inf_{\alpha\geq\alpha_{\min}} (\alpha p - \sigma(\alpha)).$$

This completes the proof. □

We deduce the following result.

Corollary 1. *Let $\zeta(p, s')$ (respectively $\sigma_{\tilde{f}_{-s'}}(\alpha)$) be as in (22) for p in the entire real line R (respectively (25) applied to $\tilde{f}_{-s'}$). If f is a periodic tempered distribution, then*

$$\forall p \in \mathbb{R} \quad \zeta(p, s') = \inf_{\alpha\in\mathbb{R}} (\alpha p - \sigma_{\tilde{f}_{-s'}}(\alpha)). \tag{30}$$

4. SOME PREVIOUS RESULTS

We will begin this section by some definitions, notations and results in Aubry and Jaffard.[19]

Definition 6. A function f is a random wavelet series if its wavelet coefficients $C_{j,k}$ defined in (12) satisfy the following requirements:

(1) $\forall\, j,k,\ C_{j,k}$ is a random variable such that $-\frac{\log(|C_{j,k}|)}{\log(2^j)}$ has law $\boldsymbol{\rho}_j$,
(2) these random variables are independent,
(3) there exists $\gamma > 0$ such that

$$\boldsymbol{\rho}(\alpha) := 1 + \inf_{\varepsilon>0} \limsup_{j\to\infty} \frac{\log(\boldsymbol{\rho}_j([\alpha-\varepsilon,\alpha+\varepsilon))}{\log(2^j)} \tag{31}$$

is strictly negative for $\alpha < \gamma$.

The function $\boldsymbol{\rho}(\alpha)$ thus defined is called the upper logarithmic density of the process. It is upper semi-continuous, but not necessarily monotonous. Note that for now we do not make any assumption on $\boldsymbol{\rho}_j$; it can be a probability measure on $\mathbb{R}\cup\{\infty\}$, $\boldsymbol{\rho}_j(\{\infty\})$ being the probability that $C_{j,k} = 0$.

Consider an arbitrary function f, which can be for instance a realization of a random wavelet series. Let $\alpha \in \mathbb{R}$, for each $j \geq 0$ let

$$N_j(\alpha) = \mathrm{Card}\{k \in \{0,\ldots,2^j-1\}; |C_{j,k}| \geq 2^{-\alpha j}\}. \tag{32}$$

Define the wavelet profile as

$$\nu(\alpha) = \limsup_{j\to\infty} \frac{\log(N_j(\alpha))}{\log(2^j)}, \tag{33}$$

and $\bar\nu$ the upper closure of ν: the hypograph of $\bar\nu$ is the closure of the hypograph of ν, or, since ν is obviously increasing

$$\bar\nu(\alpha) = \lim_{\alpha'\to\alpha^+} \nu(\alpha'). \tag{34}$$

The wavelet density $\rho(\alpha)$ is defined by

$$\rho(\alpha) = \inf_{\varepsilon>0} \rho(\alpha,\varepsilon), \tag{35}$$

where

$$\rho(\alpha,\varepsilon) = \limsup_{j\to\infty} \left(\frac{\log(N_j(\alpha+\varepsilon) - N_j(\alpha-\varepsilon))}{\log(2^j)}\right). \tag{36}$$

An heuristic interpretation is that at scale j (when $j\to\infty$), there are about $2^{\rho(\alpha)j}$ wavelet coefficients of size $|C_{j,k}|$ of order $2^{-\alpha j}$.

In Ref. 19, Aubry and Jaffard proved the following proposition.

Proposition 3. *The wavelet density $\rho(\alpha)$ defined in* (35) *is upper semi-continuous, and for all $\alpha \geq 0$, the upper closure $\bar\nu$ (defined in* (34) *of ν (defined in* (33)) *is given by*

$$\bar\nu(\alpha) = \sup_{\alpha'\leq\alpha} \rho(\alpha').$$

In the case where f is a random wavelet series, $\boldsymbol{\rho}$ is a deterministic function whereas ρ is random. The first important result of Ref. 19 links these two functions. Let

$$W := \left\{\alpha; \forall\,\varepsilon>0, \sum_{j=0}^{\infty} 2^j \boldsymbol{\rho}_j([\alpha-\varepsilon,\alpha+\varepsilon]) = \infty\right\}.$$

Clearly $\boldsymbol{\rho}(\alpha) > 0 \Rightarrow \alpha \in W$ and $\boldsymbol{\rho}(\alpha) < 0 \Rightarrow \alpha \notin W$.

The following theorem and Corollary were obtained by Aubry and Jaffard.[19]

Theorem 1. *For a random wavelet series (as in Definition* 6)*, almost surely, for all $\alpha \geq 0$,*

$$\rho(\alpha) = \begin{cases} \boldsymbol{\rho}(\alpha) & \textit{if } \alpha \in W \\ -\infty & \textit{else}. \end{cases} \tag{37}$$

Corollary 2. *A random wavelet series f (as in Definition* 6)*, almost surely, $f \in C^\gamma(\mathbb{T})$.*

For simplicity, in the rest of this paper we assume that

$$\boldsymbol{\rho}(\alpha) = 0 \Rightarrow \alpha \in W, \tag{38}$$

in which case (37) boils down to

$$\rho(\alpha) = \begin{cases} \boldsymbol{\rho}(\alpha) & \text{if } \alpha \geq 0 \\ -\infty & \text{else}. \end{cases} \tag{39}$$

Let us define

$$h_{\min} = \inf W \tag{40}$$

and

$$h_{\max} = \left(\sup_{\alpha>0} \frac{\boldsymbol{\rho}(\alpha)}{\alpha}\right)^{-1}. \tag{41}$$

Let

$$\boldsymbol{\sigma}(h) = h \sup_{0<\alpha\leq h} \frac{\boldsymbol{\rho}(\alpha)}{\alpha}. \tag{42}$$

In Ref. 19, Aubry and Jaffard obtained also the following theorem.

Theorem 2. *Let f be a random wavelet series (as in Definition* 6)*. Assume that assumption* (38) *holds. Let $\boldsymbol{\sigma}(h)$ and $\boldsymbol{\rho}(\alpha)$ are as in* (42) *and* (31)*, respectively. Almost surely, f has the following properties.*

The Hölder spectrum $d(h)$ *is defined for* $h \in [h_{\min}, h_{\max}]$. *If* $h_{\max} = 0$, *then* $d(h)$ *is reduced to* $d(0) = 1$. *If* $h_{\max} > 0$, $d(0) = 0$, *and for* $0 < h \leq h_{\max}$,

$$d(h) = \boldsymbol{\sigma}(h). \tag{43}$$

The spectrum of oscillating singularities of f *is defined on* $[h_{\min}, h_{\max}] \times [0, \infty)$, *where*

$$d(h, \beta) = (1+\beta)\boldsymbol{\rho}\left(\frac{h}{1+\beta}\right). \tag{44}$$

If $h_{\max} > 0$ *then*

$$d(h) = \sup_{\beta \geq 0} d(h, \beta). \tag{45}$$

Let us now write some definitions, notations and results in Aubry.[34] The wavelet leaders profile $\kappa(\alpha)$ is defined by

$$\kappa(\alpha) = \limsup_{j \mapsto \infty} \frac{\log(M_j(\alpha))}{\log(2^j)}. \tag{46}$$

Denote $\bar{\kappa}(\alpha)$ its upper closure: the hypograph of $\bar{\kappa}$ is the closure of the hypograph of κ, or, since κ is obviously increasing

$$\bar{\kappa}(\alpha) = \lim_{\alpha' \to \alpha^+} \kappa(\alpha'). \tag{47}$$

Remark 3. Functions κ and σ defined, respectively in (46) and (25) are the equivalent, for the wavelet leaders, of the functions ν and ρ. In particular, the like of Proposition 3 holds for them.

The following result and remark were found in Aubry.[34]

Proposition 4. *Let* f *be a random wavelet series (as in Definition* 6). *Assume that assumption* (38) *holds. Let* $\bar{\kappa}(\alpha)$, $\sigma(\alpha)$ *and* $\boldsymbol{\sigma}(h)$ *be as in* (46), (25) *and* (42), *respectively. Then almost surely,*

$$\bar{\kappa}(\alpha) = \sigma(\alpha) = \begin{cases} \boldsymbol{\sigma}(\alpha) & \text{if } \alpha \in [h_{\min}, h_{\max}] \\ -\infty & \text{else.} \end{cases}$$

Remark 4. By Theorem 2, it turns out that almost surely, these functions coincide with the Hölder spectrum $d(h)$.

5. THE OSCILLATING MULTIFRACTAL FORMALISM FOR RANDOM WAVELET SERIES

We will finally apply Proposition 2 and the results above to obtain the following theorem.

Theorem 3. *Let* f *be a random wavelet series (as in Definition* 6). *Assume that assumption* (38) *holds. Then almost surely formula* (10) *extended to* $p \in \mathbb{R}$ *holds. Further, if* $d(h, \beta)$ *is concave then almost surely the multifractal formalism of oscillating singularities* (11) *extended to* $p \in \mathbb{R}$ *holds.*

Proof of Theorem 3. Proposition 4 implies that for a random wavelet series, almost surely $K = [h_{\min}, h_{\max}]$ is compact. Therefore, applying Proposition 2, we obtain almost surely

$$\forall p \in \mathbb{R} \quad \omega(p) = \inf_{\alpha \geq 0}(\alpha p - \sigma(\alpha)).$$

Since $\tilde{f}_{-s'}$ is also a random wavelet series then almost surely

$$\forall p \in \mathbb{R} \quad \zeta(p, s') = \inf_{\alpha \geq 0}(\alpha p - \sigma_{\tilde{f}_{-s'}}(\alpha)).$$

It follows from Remark 4 that almost surely

$$\forall p \in \mathbb{R} \quad \zeta(p, s') = \inf_{\alpha \geq 0}(\alpha p - d_{\tilde{f}_{-s'}}(\alpha)).$$

On the other hand, from Theorem 2, almost surely,

$$d_{\tilde{f}_{-s'}}(\alpha) = \sup_{\beta \geq 0} d_{\tilde{f}_{-s'}}(\alpha, \beta)$$

and

$$d_{\tilde{f}_{-s'}}(\alpha, \beta) = (1+\beta)\rho_{\tilde{f}_{-s'}}\left(\frac{\alpha}{1+\beta}\right).$$

Clearly

$$\rho_{\tilde{f}_{-s'}}(\alpha) = \rho_f(\alpha + s').$$

Therefore almost surely,

$$\begin{aligned} d_{\tilde{f}_{-s'}}(\alpha, \beta) &= (1+\beta)\rho\left(\frac{\alpha}{1+\beta} + s'\right) \\ &= (1+\beta)\rho\left(\frac{\alpha + s'(1+\beta)}{1+\beta}\right) \\ &= d(\alpha + s'(1+\beta), \beta). \end{aligned}$$

Hence almost surely

$$\forall p \in \mathbb{R} \quad \zeta(p, s') = \inf_{\alpha \geq 0}(\alpha p - \sup_{\beta \geq 0} d(\alpha + s'(1+\beta), \beta)).$$

Whence almost surely

$$\forall p \in \mathbb{R} \quad \zeta(p, s') = \inf_{\alpha \geq 0} \inf_{\beta \geq 0}(\alpha p - d(\alpha + s'(1+\beta), \beta)).$$

By a change of variables we get almost surely (10) extended to $p \in \mathbb{R}$. It follows that for a random wavelet series almost surely the function $(p, s') \mapsto \zeta(p, s')$ is concave. So, if $d(h, \beta)$ is concave then almost surely (11) extended to $p \in \mathbb{R}$ holds. □

6. CONCLUSION

The scaling function $\zeta(p, s')$ takes into account strong correlations between the positions of large wavelet coefficients. This requirement is backed by several numerical studies which have uncovered such correlations in turbulence,[35] image processing,[36] traffic,[37] finance,[38] etc. Several random processes have been introduced in order to model such data; a particularly important class is supplied by multiplicative models, see, for instance, Refs. 21, 39, 40, 41 and 42 and references therein.

In this paper, using some results from Jaffard *et al.*,[25] we extended the scaling function to $p \in \mathbb{R}$ and deduced its concavity with respect to $p \in \mathbb{R}$ and independence on the orthonormal wavelet basis in the Schwartz class. We also proved its concavity with respect to s' when $p > 0$. We then proved a result which relates $\omega(p)$ (for $p \in \mathbb{R}$) to the wavelet leaders density $\sigma(\alpha)$. We therefore deduced a result which relates $\zeta(p, s')$ for $p \in \mathbb{R}$ to $\sigma_{f_{-s'}}(\alpha)$. As an application, we used the generalized scaling function $\zeta(p, s')$ to study the validity of the multifractal formalism of oscillating singularities for random wavelet series (under the assumption of independence and laws depending only on the scale). We therefore extended the results obtained by Jaffard (for lacunary wavelet series) in the general settings of random wavelet series and $p \in \mathbb{R}$.

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