Bayes' Rule:

Definition:

The events $A_1, A_2, \ldots, A_n$ constitute a partition of the sample space S if

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n = S$$

$$A_i \cap A_j = \phi, \quad \forall i \neq j$$

Theorem: (Total Probability)

If the events $A_1, A_2, \ldots, A_n$ constitute a partition of the sample space S such that $P(A_k) \neq 0$ for $k = 1, \ldots, n$ then for any even B

$$P(B) = \sum_{k=1}^{n} P(A_k)\, P(B \backslash A_k)$$

$$= \sum_{k=1}^{n} P(A_k \cap B)$$

Example:

Three machines A_1, A_2, and A_3 make 20%, 30% and 50% respectively of the products. It is known that 1%, 4% and 7% of the products made by each machine, respectively, are defective. If a finished product is randomly selected. what is the probability that it is defective?

solution

Define the following events:

$B = \{$ the selected product is defective $\}$

$A_1 = \{$ the selected product is made by machine $A_1 \}$

$A_2 = \{$ " " " " " " " " $A_2 \}$

$A_3 = \{$ " " " " " " " " $A_3 \}$

$P(A_1) = \frac{20}{100} = 0.2$, $P(B \mid A_1) = \frac{1}{100} = 0.01$

$P(A_2) = \frac{30}{100} = 0.3$, $P(B \mid A_2) = \frac{4}{100} = 0.04$

$P(A_3) = \frac{50}{100} = 0.5$, $P(B \mid A_3) = \frac{7}{100} = 0.07$

$$P(B) = \sum_{k=1}^{3} P(A_k) P(B \mid A_k)$$

$$P(A_1) P(B \mid A_1) + P(A_2) P(B \mid A_2) + P(A_3) P(B \mid A_3)$$

$$= 0.2 \times 0.01 + 0.3 \times 0.04 + 0.5 \times 0.07$$

$$= 0.002 + 0.012 + 0.035$$

$$= 0.049$$

Question:

If it is known that the selected product is defective what is the probability that it is made by machine A_1?

$$P(A_1 \mid B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{0.2 \times 0.01}{0.049} = \frac{0.002}{0.049} = 0.0408$$

Theorem: (Bayes' rule)

If the events A_1, A_2 — and A_n constitute a partition of the sample space S such that $P(A_k) \neq 0$ for $k = 1, 2, \ldots, n$ then for any event B such that $P(B) \neq 0$

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)\,P(B \mid A_i)}{\sum_{k=1}^{n} P(A_k)\,P(B \mid A_k)} = \frac{P(A_i)\,P(B \mid A_i)}{P(B)}$$

for $i = 1, 2, \ldots, n$

Example:-

In Example 2.38, if it is known that the selected product is defective. what is the probability that it is made by:

(a) machine A_2?

(b) " A_3?

(a) $$P(A_2 \mid B) = \frac{P(A_2)\,P(B \mid A_2)}{\sum P(A_k)\,P(B \mid A_k)} = \frac{P(A_2)\,P(B \mid A_2)}{P(B)}$$

$$= \frac{0.3 \times 0.04}{0.049} = \frac{0.012}{0.049}$$

$$= 0.2449$$

(b) $$P(A_3 \mid B) = \frac{P(A_3)\,P(B \mid A_3)}{\sum P(A_k)\,P(B \mid A_k)} = \frac{P(A_3)\,P(B \mid A_3)}{P(B)}$$

$$= \frac{0.5 \times 0.07}{0.049} = \frac{0.035}{0.049} = 0.7142$$

Note:

$P(A_1|B) = 0.0408$, $P(A_2|B) = 0.2449$, $p(A_3|B) = 0.7142$

$$\sum_{k=1}^{3} P(A_k|B) = 1$$

If the selected product was found defective, we should check machine A_3, if it is ok, we should check machine A_2, if it is ok, we should check machine A_1.