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Research Article

Asymptotic Study of the 2D-DQGE Solutions

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We study the regularity of the solutions of the surface quasi-geostrophic equation with subcritical exponent $1/2 < \alpha \le 1$. We prove that if the initial data is small enough in the critical space $\dot{H}^{2-2\alpha}(\mathbb{R}^2)$, then the regularity of the solution is of exponential growth type with respect to time and its $\dot{H}^{2-2\alpha}(\mathbb{R}^2)$ norm decays exponentially fast. It becomes then infinitely differentiable with respect to time and has value in all homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^2)$ for $s \ge 2 - 2\alpha$. Moreover, we give some general properties of the global solutions.

1. Introduction

We consider the $2D$ dissipative quasi-geostrophic equation with subcritical exponent $1/2 < \alpha \le 1$:

$$\begin{aligned} &\partial_t\theta + (-\Delta)^{\alpha}\theta + (u\cdot\nabla)\theta = 0 \quad \text{in } \mathbb{R}^+\times\mathbb{R}^2, \\ &\theta(0,x) = \theta^0(x) \quad \text{in } \mathbb{R}^2, \end{aligned} \qquad (\mathcal{S}_\alpha)$$

where $x \in \mathbb{R}^2$, $t > 0$, $\theta = \theta(x,t)$ is the unknown potential temperature, and $u = (u_1, u_2)$ is the divergence free velocity which is determined by the Riesz transformation of θ in the following sense:

$$\begin{aligned} u_1 &= -\mathcal{R}_2\theta = -\partial_2(-\Delta)^{-1/2}\theta, \\ u_2 &= \mathcal{R}_1\theta = \partial_1(-\Delta)^{-1/2}\theta. \end{aligned} \qquad (1)$$

The critical homogeneous Sobolev space is $\dot{H}^{2-2\alpha}(\mathbb{R}^2)$ and we have

$$\left\|\lambda^{2\alpha-1} f(\lambda .)\right\|_{\dot{H}^{2-2\alpha}} = \|f\|_{\dot{H}^{2-2\alpha}}, \quad \forall\lambda > 0. \qquad (2)$$

In [1], we studied the existence of global solutions of $(\mathcal{S}_\alpha)$ if the initial data θ^0 is small in the critical space $\dot{H}^{2-2\alpha}(\mathbb{R}^2)$ and the subcritical exponent $\alpha \in (1/2, 1]$. In use of Theorem 4.2 in [2] with $p = q = 2$, we proved the following Theorem.

Theorem 1 (see [1]). *For $\alpha \in (1/2, 1]$ and $\theta^0 \in \dot{H}^{2-2\alpha}(\mathbb{R}^2)$, there exists a constant $c_\alpha > 0$ such that if*

$$\left\|\theta^0\right\|_{\dot{H}^{2-2\alpha}} < c_\alpha, \qquad (3)$$

the initial value problem $(\mathcal{S}_\alpha)$ has a unique solution in $\mathcal{C}_b(\mathbb{R}^+, \dot{H}^{2-2\alpha}(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+, \dot{H}^{2-\alpha}(\mathbb{R}^2))$. Moreover,

$$\|\theta(t)\|^2_{\dot{H}^{2-2\alpha}} + \int_0^t \|\theta(\tau)\|^2_{\dot{H}^{2-\alpha}} d\tau \le \left\|\theta^0\right\|^2_{\dot{H}^{2-2\alpha}}, \quad \forall t \ge 0. \qquad (4)$$

We proved also the following result.

Theorem 2 (see [1]). *Let $2/3 \le \alpha \le 1$.*

(i) *If $\theta \in \mathcal{C}(\mathbb{R}^+, \dot{H}^{2-2\alpha}(\mathbb{R}^2))$ is a global solution of $(\mathcal{S}_\alpha)$, then*

$$\lim_{t\to\infty} \|\theta(t)\|_{\dot{H}^{2-2\alpha}} = 0. \qquad (5)$$

(ii) *If $\theta \in \mathcal{C}(\mathbb{R}^+, H^{2-2\alpha}(\mathbb{R}^2))$ is a global solution of $(\mathcal{S}_\alpha)$, then*

$$\lim_{t\to\infty} \|\theta(t)\|_{H^{2-2\alpha}} = 0. \qquad (6)$$

In this paper, we describe the long time behavior of these solutions with respect to the homogeneous Sobolev norm $\|\cdot\|_{\dot{H}^s}$, for $s \ge 2 - 2\alpha$. We prove the following.

Theorem 3. *There exists $c_\alpha > 0$ such that, for all $\theta^0 \in \dot{H}^{2-2\alpha}(\mathbb{R}^2)$, $\|\theta^0\|_{\dot{H}^{2-2\alpha}} < c_\alpha$, and there exists a global solution $\theta \in \mathscr{C}_b(\mathbb{R}^+, \dot{H}^{2-2\alpha}(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+, \dot{H}^{2-\alpha}(\mathbb{R}^2))$ such that, for all $s > 2 - 2\alpha$, $\theta(t) \in \dot{H}^s(\mathbb{R}^2)$ for all $t > 0$, and*

$$\|\theta(t)\|_{\dot{H}^s} = O\left(t^{-(s-(2-2\alpha))/2\alpha}\right), \quad t \longrightarrow \infty. \tag{7}$$

When the initial data is in $\dot{H}^{2-2\alpha}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and small enough in the homogeneous space $\dot{H}^{2-2\alpha}(\mathbb{R}^2)$, we prove that the Leray solution is also in all Sobolev spaces $\dot{H}^s(\mathbb{R}^2)$. Moreover, we describe the long time behavior of its homogeneous Sobolev norm $\|\cdot\|_{\dot{H}^s}$, for $s \geq 0$. We state also the following.

Theorem 4. *There exists $c_\alpha > 0$ such that, for all $\theta^0 \in H^{2-2\alpha}(\mathbb{R}^2)$, $\|\theta^0\|_{\dot{H}^{2-2\alpha}} < c_\alpha$, and there exists a global solution $\theta \in \mathscr{C}_b(\mathbb{R}^+, H^{2-2\alpha}(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+, \dot{H}^{2-\alpha}(\mathbb{R}^2))$ such that, for all $s \geq 0$,*

$$\|\theta(t)\|_{\dot{H}^s} = O\left(t^{-s/2\alpha}\right), \quad t \longrightarrow \infty. \tag{8}$$

The paper is organized as follows. We start by recalling some preliminary background and stating useful preliminary results on Sobolev spaces. Sections 3 and 4 are devoted to the proof of the main results, Theorems 3 and 4. In Section 5, we give some general properties for any global solutions of the system $(\mathcal{S}_\alpha)$.

2. Notations and Preliminary Results

2.1. Notations and Technical Lemmas. In this short section, we collect some notations and definitions that will be used later and we give some technical lemmas.

(i) The Fourier transformation in $\mathbb{R}^2$ is normalized as

$$\mathscr{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^2} \exp(-ix\cdot\xi) f(x)\,dx, \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2. \tag{9}$$

(ii) The inverse Fourier formula is

$$\mathscr{F}^{-1}(g)(x) = (2\pi)^{-2} \int \exp(i\xi\cdot x) f(\xi)\,d\xi, \quad x = (x_1, x_2) \in \mathbb{R}^2. \tag{10}$$

(iii) For $s \in \mathbb{R}$, $H^s(\mathbb{R}^2)$ denotes the usual nonhomogeneous Sobolev space on $\mathbb{R}^2$ and $\langle\cdot,\cdot\rangle_{H^s}$ its scalar product.

(iv) For $s \in \mathbb{R}$, $\dot{H}^s(\mathbb{R}^2)$ denotes the usual homogeneous Sobolev space on $\mathbb{R}^2$ and $\langle\cdot,\cdot\rangle_{\dot{H}^s}$ its scalar product.

(v) The convolution product of a suitable pair of functions f and g on $\mathbb{R}^2$ is given by

$$(f * g)(x) := \int_{\mathbb{R}^2} f(y)\, g(x-y)\,dy. \tag{11}$$

(vi) For any Banach space $(B, \|\cdot\|)$, any real number $1 \leq p \leq \infty$, and any time $T > 0$, we will denote by $L^p_T(B)$ the space of measurable functions $t \in [0,T] \mapsto f(t) \in B$ such that $(t \mapsto \|f(t)\|) \in L^p([0,T])$.

(vii) If $f = (f_1, f_2)$ and $g = (g_1, g_2)$ are two vector fields, we set

$$f \otimes g := (g_1 f, g_2 f), \quad \operatorname{div}(f\otimes g) := (\operatorname{div}(g_1 f), \operatorname{div}(g_2 f)). \tag{12}$$

(viii) For any subset X of a set E, $\mathbf{1}_X$ denotes the characteristic function of X.

We recall a fundamental lemma concerning some product laws in homogeneous Sobolev spaces.

Lemma 5 (see [3]). *Let s_1, s_2 be two real numbers such that*

$$s_1 < 1, \qquad s_1 + s_2 > 0. \tag{13}$$

There exists a constant $C := C(s_1, s_2)$, such that, for all $f, g \in \dot{H}^{s_1}(\mathbb{R}^2) \cap \dot{H}^{s_2}(\mathbb{R}^2)$,

$$\|fg\|_{\dot{H}^{s_1+s_2-1}} \leq C\left(\|f\|_{\dot{H}^{s_1}}\|g\|_{\dot{H}^{s_2}} + \|f\|_{\dot{H}^{s_2}}\|g\|_{\dot{H}^{s_1}}\right). \tag{14}$$

If $s_1, s_2 < 1$ and $s_1 + s_2 > 0$, there exists a constant $c = c(s_1, s_2)$, such that, for all $f \in \dot{H}^{s_1}(\mathbb{R}^2)$ and $g \in \dot{H}^{s_2}(\mathbb{R}^2)$,

$$\|fg\|_{\dot{H}^{s_1+s_2-1}} \leq c\|f\|_{\dot{H}^{s_1}}\|g\|_{\dot{H}^{s_2}}. \tag{15}$$

For the proof of the main results, we need the following lemma.

Lemma 6. *Under the same conditions as in Theorem 3, for all $\sigma \geq 0$ and $\varepsilon \geq 0$,*

$$\int_{\mathbb{R}^2} |\xi|^{2\sigma} e^{2\varepsilon t|\xi|} |\mathscr{F}((u\cdot\nabla)\theta)\mathscr{F}(w)|\,d\xi \leq C\|\theta_\varepsilon\|_{\dot{H}^{2-2\alpha}}\|\theta_\varepsilon\|_{\dot{H}^{\sigma+\alpha}}\|w_\varepsilon\|_{\dot{H}^{\sigma+\alpha}}, \tag{16}$$

where $\mathscr{F}(\theta_\varepsilon) = e^{2\varepsilon t|\xi|}\mathscr{F}(\theta)$ and $\mathscr{F}(w_\varepsilon) = e^{2\varepsilon t|\xi|}\mathscr{F}(w)(t,\xi)$ and $w_\varepsilon \in \dot{H}^{\sigma+\alpha}(\mathbb{R}^2)$.

Remarks 7. (i) If $\sigma = 0$, formula (16) gives

$$\int_{\mathbb{R}^2} e^{2\varepsilon t|\xi|} |\mathscr{F}((u\cdot\nabla)\theta)\widehat{w}|\,d\xi \leq C\|\theta_\varepsilon\|_{\dot{H}^{2-2\alpha}}\|\theta_\varepsilon\|_{\dot{H}^{\alpha}}\|w_\varepsilon\|_{\dot{H}^{\alpha}}. \tag{17}$$

(ii) If $\varepsilon = 0$ and $\sigma = 0$, formula (16) gives

$$\int_{\mathbb{R}^2} |\mathscr{F}((u\cdot\nabla)\theta)\widehat{w}|\,d\xi \leq C\|\theta\|_{\dot{H}^{2-2\alpha}}\|\theta\|_{\dot{H}^{\alpha}}\|w\|_{\dot{H}^{\alpha}}. \tag{18}$$

(iii) If $\varepsilon = 0$ and $\sigma = 2 - 2\alpha$, formula (16) gives

$$\int_{\mathbb{R}^2} |\xi|^{2(2-2\alpha)} e^{2\varepsilon t|\xi|} |\mathscr{F}((u\cdot\nabla)\theta)\widehat{w}|\,d\xi \leq C\|\theta\|_{\dot{H}^{2-2\alpha}}\|\theta\|_{\dot{H}^{2-\alpha}}\|w\|_{\dot{H}^{2-\alpha}}. \tag{19}$$

Proof of Lemma 6. By the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\int |\xi|^{2\sigma} e^{2\varepsilon t|\xi|} |\mathscr{F}((u\cdot\nabla)\theta)\widehat{w}|\, d\xi \\
&\le \int |\xi|^{\sigma-\alpha} e^{\varepsilon t|\xi|} |\mathscr{F}((u\cdot\nabla)\theta)|\, e^{\varepsilon t|\xi|}|\xi|^{\sigma+\alpha} |\widehat{w}|\, d\xi \\
&\le \left(\int |\xi|^{2(\sigma-\alpha)} e^{2\varepsilon t|\xi|} |\mathscr{F}((u\cdot\nabla)\theta)|^2 d\xi\right)^{1/2} \|w_\varepsilon\|_{\dot H^{\sigma+\alpha}}.
\end{aligned} \tag{20}$$

Using the weak derivatives properties, the elementary inequality $e^{a|\xi|} \le e^{a|\xi-\eta|}e^{a|\xi|}$, with $a \ge 0$ and $\xi, \eta \in \mathbb{R}^2$, and the product laws (Lemma 5), with $s_1 + s_2 = \sigma - \alpha + 2 > 0$, $s_1 = 2-2\alpha < 1$, and $s_2 = \sigma+\alpha$, we can dominate the nonlinear part of (20) as follows:

$$\begin{aligned}
&\int |\xi|^{2(\sigma-\alpha)} e^{2\varepsilon t|\xi|} |\mathscr{F}((u\cdot\nabla)\theta)|^2 d\xi \\
&\le \int |\xi|^{2(\sigma-\alpha+1)} e^{2\varepsilon t|\xi|} \left(|\widehat{\theta}| * |\widehat{\theta}|\right)^2 d\xi \\
&\le \int |\xi|^{2(\sigma-\alpha+1)} \cdot \left(e^{2\varepsilon t|\cdot|}|\widehat{\theta}| * e^{2\varepsilon t|\cdot|}|\widehat{\theta}|\right)^2 d\xi \\
&\le C\|\theta_\varepsilon\|^2_{\dot H^{2-2\alpha}} \|\theta_\varepsilon\|^2_{\dot H^{\sigma+\alpha}}.
\end{aligned} \tag{21}$$

□

3. Proof of Theorem 3

To prove Theorem 3, we need the following result.

Proposition 8. *There exists $c_\alpha > 0$ such that, for all $\theta^0 \in \dot H^{2-2\alpha}(\mathbb{R}^2)$, $\|\theta^0\|_{\dot H^{2-2\alpha}(\mathbb{R}^2)} < c_\alpha$, and there exists a global solution $\theta \in \mathscr{C}_b(\mathbb{R}^+, \dot H^{2-2\alpha}(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+, \dot H^{2-\alpha}(\mathbb{R}^2))$ such that*

$$\begin{aligned}
&\int e^{t^{1/2\alpha}|\xi|} |\xi|^{2(2-2\alpha)} \left|\widehat{\theta}(t,\xi)\right|^2 d\xi \\
&\quad + \int_0^t \int e^{\tau t^{-1+1/2\alpha}|\xi|} |\xi|^{2(2-\alpha)} \left|\widehat{\theta}(\tau,\xi)\right|^2 d\xi\, d\tau \\
&\le 2\left\|\theta^0\right\|^2_{\dot H^{2-2\alpha}}.
\end{aligned} \tag{22}$$

Proof. The proof is done in two steps.

First Step. For a nonnegative integer n, Friedrich's operator J_n is defined by

$$J_n(f) := \mathscr{F}^{-1}\left(\mathbf{1}_{\{1/n<|\xi|<n\}}\widehat{f}\right). \tag{23}$$

Consider the following approximate system $(\mathscr{S}^n_\alpha)$ on $\mathbb{R}_+ \times \mathbb{R}^2$:

$$\begin{aligned}
&\partial_t\theta + (-\Delta)^\alpha J_n\theta + J_n\left(J_n u \cdot \nabla J_n\theta\right) = 0, \\
&u = (-\Delta)^{-1/2}\left(-\partial_2\theta, \partial_1\theta\right), \\
&\theta \mid t = 0 = J_n\theta^0.
\end{aligned} \tag{24}$$

Then, by the ordinary differential equations theory, the system $(\mathscr{S}^n_\alpha)$ has a unique maximal solution θ_n in the space $\mathscr{C}^1([0,T^*_n), L^2(\mathbb{R}^2))$, $T^*_n > 0$. Using the uniqueness and the fact that $J_n^2 = J_n$, we obtain $J_n\theta_n = \theta_n$ and

$$\begin{aligned}
&\partial_t\theta_n + (-\Delta)^\alpha\theta_n + J_n\left(u_n\cdot\nabla\theta_n\right) = 0, \\
&u_n = (-\Delta)^{-1/2}\left(-\partial_2\theta_n, \partial_1\theta_n\right), \\
&\theta_n \mid t = 0 = J_n\theta^0.
\end{aligned} \tag{25}$$

Taking the scalar product in $L^2(\mathbb{R}^2)$, we obtain, for $t \in [0, T^*_n)$,

$$\partial_t\|\theta_n\|^2_{L^2} + 2\|\theta_n\|^2_{\dot H^\alpha} \le 0. \tag{26}$$

It follows that, for all $t \in [0,T^*_n)$, $\|\theta_n(t)\|^2_{L^2} \le \|J_n\theta^0\|^2_{L^2}$, which implies that $T^*_n = +\infty$.

Now, taking scalar product in $\dot H^{2-2\alpha}(\mathbb{R}^2)$, we obtain

$$\begin{aligned}
&\partial_t\|\theta_n\|^2_{\dot H^{2-2\alpha}} + 2\|\theta_n\|^2_{\dot H^{2-\alpha}} \\
&\quad \le 2\left|\left\langle J_n\left(u_n\cdot\nabla\theta_n\right), \theta_n\right\rangle_{\dot H^{2-2\alpha}}\right|, \\
&\left|\left\langle J_n\left(u_n\cdot\nabla\theta_n\right), \theta_n\right\rangle_{\dot H^{2-2\alpha}}\right| \\
&\quad = \left|\left\langle u_n\cdot\nabla\theta_n, J_n\theta_n\right\rangle_{\dot H^{2-2\alpha}}\right| \\
&\quad = \left|\left\langle \operatorname{div}\left(\theta_n u_n\right), \theta_n\right\rangle_{\dot H^{2-2\alpha}}\right| \\
&\quad \le \left\|\operatorname{div}(\theta_n u_n)\right\|_{\dot H^{2-\alpha}}\left\|\theta_n\right\|_{\dot H^{2-\alpha}} \\
&\quad \le \left\|\theta_n u_n\right\|_{\dot H^{2-3\alpha+1}}\left\|\theta_n\right\|_{\dot H^{2-\alpha}}.
\end{aligned} \tag{27}$$

Using product law (15) with $s_1 = 2-2\alpha < 1$ and $s_2 = 2-\alpha$, we obtain

$$\begin{aligned}
&\left|\left\langle J_n\left(u_n\cdot\nabla\theta_n\right), \theta_n\right\rangle_{\dot H^{2-2\alpha}}\right| \\
&\quad \le c(\alpha)\left(\|\theta_n\|_{\dot H^{2-2\alpha}}\|u_n\|_{\dot H^{2-\alpha}} + \|\theta_n\|_{\dot H^{2-\alpha}}\|u_n\|_{\dot H^{2-2\alpha}}\right)\|\theta_n\|_{\dot H^{2-\alpha}}.
\end{aligned} \tag{28}$$

But

$$\begin{aligned}
&\|u_n\|^2_{\dot H^s} = \int |\xi|^{2s}\left|\widehat{\theta_n}(t,\xi)\right|^2 \left|\left(\frac{i\xi_2}{|\xi|}, \frac{-i\xi_1}{|\xi|}\right)\right|^2 d\xi = \|\theta_n\|^2_{\dot H^s}, \\
&\left|\left\langle J_n(u_n\cdot\nabla\theta_n), \theta_n\right\rangle_{\dot H^{2-2\alpha}}\right| \le c(\alpha)\|\theta_n\|_{\dot H^{2-2\alpha}}\|\theta_n\|^2_{\dot H^{2-\alpha}}.
\end{aligned} \tag{29}$$

Then,

$$\partial_t\|\theta_n\|^2_{\dot H^{2-2\alpha}} + 2\|\theta_n\|^2_{\dot H^{2-\alpha}} \le c(\alpha)\|\theta_n\|_{\dot H^{2-2\alpha}}\|\theta_n\|^2_{\dot H^{2-\alpha}}. \tag{30}$$

Let

$$T_n := \sup\left\{t \ge 0, \|\theta_n\|_{L^\infty_t(\dot H^{2-2\alpha})} < 2c_\alpha\right\}. \tag{31}$$

For $0 \le t < T_n$, by (30), we have

$$\begin{aligned}
&\|\theta_n(t)\|^2_{\dot H^{2-2\alpha}} + 2\int_0^t \|\theta_n\|^2_{\dot H^{2-\alpha}} \le \left\|\theta^0\right\|^2_{\dot H^{2-2\alpha}} + \int_0^t \|\theta_n\|^2_{\dot H^{2-\alpha}} d\tau; \\
&\|\theta_n(t)\|^2_{\dot H^{2-2\alpha}} + \int_0^t \|\theta_n\|^2_{\dot H^{2-\alpha}} \le \left\|\theta^0\right\|^2_{\dot H^{2-2\alpha}} < c_\alpha^2,
\end{aligned} \tag{32}$$

then $T_n = \infty$, and, for all $t \geq 0$, we have

$$\|\theta_n(t)\|^2_{\dot{H}^{2-2\alpha}} + \int_0^t \|\theta_n\|^2_{\dot{H}^{2-\alpha}} \leq \|\theta^0\|^2_{\dot{H}^{2-2\alpha}}. \tag{33}$$

If we take the limit when n goes to the infinity, we find a solution $\theta \in \mathscr{C}_b(\mathbb{R}^+, \dot{H}^{2-2\alpha}(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+, \dot{H}^{2-\alpha}(\mathbb{R}^2))$ which satisfies

$$\|\theta(t)\|^2_{\dot{H}^{2-2\alpha}} + \int_0^t \|\theta\|^2_{\dot{H}^{2-\alpha}} \leq \|\theta^0\|^2_{\dot{H}^{2-2\alpha}}, \tag{34}$$

which proves the first result of Theorem 3.

Second Step. Back to the approximate system,

$$\partial_t\widehat{\theta_n} + |\xi|^{2\alpha}\widehat{\theta_n} + \mathbf{1}_{1/n<|\xi|<n}\mathscr{F}(u^n \cdot \nabla\theta_n) = 0,$$
$$\partial_t\left|\widehat{\theta_n}\right|^2 + 2|\xi|^{2\alpha}\left|\widehat{\theta_n}\right|^2 + 2\,\mathrm{Re}\left(\mathscr{F}(u^n \cdot \nabla\theta_n)(\xi) \cdot \widehat{\theta_n}(-\xi)\right) = 0. \tag{35}$$

For $\varepsilon > 0$, we define

$$f_n = f_{n,\varepsilon} := \mathscr{F}^{-1}\left(e^{\varepsilon t|\xi|}\widehat{\theta_n}\right),$$
$$\widehat{f_n} = e^{\varepsilon t|\xi|}\widehat{\theta_n}. \tag{36}$$

Then,

$$\begin{aligned}
&\partial_t\left|\widehat{f_n}\right|^2 + 2|\xi|^{2\alpha}\left|\widehat{f_n}\right|^2 \\
&= 2\varepsilon\left|\widehat{f_n}\right|^2 - 2\,\mathrm{Re}\,e^{2\varepsilon t|\xi|}\left(\mathscr{F}(u^n \cdot \nabla\theta_n)(\xi) \cdot \widehat{\theta_n}(-\xi)\right) \\
&= 2\varepsilon|\xi| \cdot \left|\widehat{f_n}\right|^2 - 2\,\mathrm{Re}\left(e^{\varepsilon t|\xi|}\mathscr{F}(u^n \cdot \nabla\theta_n)(\xi) \cdot e^{\varepsilon t|\xi|}\widehat{\theta_n}(-\xi)\right) \\
&= 2\varepsilon|\xi| \cdot \left|\widehat{f_n}\right|^2 - 2\,\mathrm{Re}\left(e^{\varepsilon t|\xi|}\mathscr{F}(u^n \cdot \nabla\theta_n)(\xi) \cdot \widehat{f_n}(-\xi)\right) \\
&= 2\varepsilon|\xi| \cdot \left|\widehat{f_n}\right|^2 - 2\,\mathrm{Re}\left(e^{\varepsilon t|\xi|}\mathscr{F}(\mathrm{div}(\theta_n u^n))(\xi) \cdot \widehat{f_n}(-\xi)\right) \\
&\leq 2\varepsilon|\xi| \cdot \left|\widehat{f_n}\right|^2 + 2e^{\varepsilon t|\xi|}|\xi| \cdot \left|\mathscr{F}(\theta_n u^n)\right| \cdot \left|\widehat{f_n}(\xi)\right| \\
&\leq 2\varepsilon|\xi| \cdot \left|\widehat{f_n}\right|^2 + 2e^{\varepsilon t|\xi|}|\xi| \cdot \left(\left|\widehat{\theta_n}\right| * \left|\widehat{u^n}\right|\right) \cdot \left|\widehat{f_n}(\xi)\right| \\
&\leq 2\varepsilon|\xi| \cdot \left|\widehat{f_n}\right|^2 + 2e^{\varepsilon t|\xi|}|\xi| \cdot \left(\left|\widehat{\theta_n}\right| * \left|\widehat{\theta_n}\right|\right) \cdot \left|\widehat{f_n}(\xi)\right|.
\end{aligned} \tag{37}$$

Using the classical inequality

$$e^{\lambda|\xi|} \leq e^{\lambda|\xi-\eta|}e^{\lambda|\eta|}, \quad \forall\lambda \geq 0, \tag{38}$$

we let

$$\partial_t\left|\widehat{f_n}\right|^2 + 2|\xi|^{2\alpha}\left|\widehat{f_n}\right|^2 \leq 2\varepsilon|\xi| \cdot \left|\widehat{f_n}\right|^2 + 2|\xi|\left(\left|\widehat{f_n}\right| * \left|\widehat{f_n}\right|\right) \cdot \left|\widehat{f_n}\right|. \tag{39}$$

Taking the norm in $\dot{H}^{2-2\alpha}$, we obtain

$$\begin{aligned}
&\partial_t\|f_n\|^2_{\dot{H}^{2-2\alpha}} + 2\|f_n\|^2_{\dot{H}^{2-\alpha}} \\
&\leq 2\varepsilon\|f_n\|^2_{\dot{H}^{2-2\alpha+1/2}} + 2\int|\xi|^{2(2-2\alpha)+1}\left(\left|\widehat{f_n}\right| * \left|\widehat{f_n}\right|\right) \cdot \left|\widehat{f_n}\right| d\xi \\
&\leq 2\varepsilon\|f_n\|^2_{\dot{H}^{(2-2\alpha)+1/2}} \\
&\quad + 2\int|\xi|^{(2-2\alpha)+1-\alpha}\left(\left|\widehat{f_n}\right| * \left|\widehat{f_n}\right|\right)|\xi|^{2-\alpha}\left|\widehat{f_n}\right| d\xi.
\end{aligned} \tag{40}$$

By Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\partial_t\|f_n\|^2_{\dot{H}^{2-2\alpha}} + 2\|f_n\|^2_{\dot{H}^{2-\alpha}} \\
&\leq 2\varepsilon\|f_n\|^2_{\dot{H}^{(2-2\alpha)+1/2}} \\
&\quad + 2\left[\int|\xi|^{2((2-2\alpha)+1-\alpha)}\left(\left|\widehat{f_n}\right| * \left|\widehat{f_n}\right|\right) d\xi\right]^2\|f_n\|_{\dot{H}^{s_c+\alpha}} \\
&\leq 2\varepsilon\|f_n\|^2_{\dot{H}^{(2-2\alpha)+1/2}} + 2\|g_n \cdot g_n\|_{\dot{H}^{(2-2\alpha)+1-\alpha}}\|f_n\|_{\dot{H}^{2-\alpha}},
\end{aligned} \tag{41}$$

where

$$g_n := \mathscr{F}^{-1}\left(\left|\widehat{f_n}\right|\right). \tag{42}$$

Using product law (15) in the homogeneous Sobolev space with $s_1 = 2-2\alpha < 1$, $s_2 = 2-\alpha$, and $s_1+s_2 = (2-2\alpha)+(2-\alpha) > 0$, we obtain

$$\begin{aligned}
&\partial_t\|f_n\|^2_{\dot{H}^2-2\alpha} + 2\|f_n\|^2_{\dot{H}^{2-\alpha}} \\
&\leq 2\varepsilon\|f_n\|^2_{\dot{H}^{(2-2\alpha)+1/2}} + C_\alpha\|g_n\|_{\dot{H}^{2-2\alpha}}\|g_n\|_{\dot{H}^{2-\alpha}}\|f_n\|_{\dot{H}^{2-\alpha}}.
\end{aligned} \tag{43}$$

Then,

$$\begin{aligned}
&\partial_t\|f_n\|^2_{\dot{H}^{2-2\alpha}} + 2\|f_n\|^2_{\dot{H}^{2-\alpha}} \\
&\leq 2\varepsilon\|f_n\|^2_{\dot{H}^{(2-2\alpha)+1/2}} + C_\alpha\|f_n\|_{\dot{H}^{2-2\alpha}}\|f_n\|^2_{\dot{H}^{2-\alpha}}.
\end{aligned} \tag{44}$$

To estimate the term $\|f_n\|^2_{\dot{H}^{(2-2\alpha)+1/2}}$, we use the Hölder inequality and we get

$$\|f_n\|^2_{\dot{H}^{(2-2\alpha)+1/2}} \leq \|f_n\|^{2(1-1/2\alpha)}_{\dot{H}^{2-2\alpha}}\|f_n\|^{1/\alpha}_{\dot{H}^{2-\alpha}}. \tag{45}$$

The convex inequality $ab \leq a^p/p + b^q/q$, with $p = 2\alpha/(2\alpha-1)$ and $q = 2\alpha$, gives

$$\varepsilon\|f_n\|^2_{\dot{H}^{(2-2\alpha)+1/2}} \leq C_\alpha\varepsilon^{2\alpha/(2\alpha-1)}\|f_n\|^2_{\dot{H}^{2-2\alpha}} + \|f_n\|^2_{\dot{H}^{2-\alpha}}. \tag{46}$$

Thus,

$$\begin{aligned}
&\partial_t\|f_n\|^2_{\dot{H}^{2-2\alpha}} + \frac{3}{2}\|f_n\|^2_{\dot{H}^{2-\alpha}} \\
&\leq C_\alpha\varepsilon^{2\alpha/(2\alpha-1)}\|f_n\|^2_{\dot{H}^{2-2\alpha}} + C_\alpha\|f_n\|_{\dot{H}^{2-2\alpha}}\|f_n\|^2_{\dot{H}^{2-\alpha}},
\end{aligned} \tag{47}$$

where f_n is in $\mathscr{C}(\mathbb{R}^+, \dot{H}^{2-2\alpha}(\mathbb{R}^2))$ and $\|f_n(0)\|_{\dot{H}^{2-2\alpha}} = \|\theta_n(0)\|_{\dot{H}^{2-2\alpha}} \leq \|\theta^0\|_{\dot{H}^{2-2\alpha}} < c_\alpha$.

Let $T > 0$ and $\varepsilon = (\ln(2)/C_\alpha T)^{(2\alpha-1)/2\alpha}$; we set

$$T_n := \sup\left\{t \geq 0; \|f_n\|_{L_t^\infty(\dot{H}^{2-2\alpha})} < 2c_\alpha\right\}. \tag{48}$$

For $0 \leq t < \inf(T, T_n)$, we have

$$\partial_t \|f_n\|_{\dot{H}^{2-2\alpha}}^2 + \|f_n\|_{\dot{H}^{2-\alpha}}^2 \leq C_\alpha \varepsilon^{2\alpha/(2\alpha-1)} \|f_n\|_{\dot{H}^{2-2\alpha}}^2. \tag{49}$$

By Gronwall lemma, we get that, for all $t \in [0, \inf(T, T_n))$,

$$\begin{aligned}\|f_n(t)\|_{\dot{H}^{2-2\alpha}}^2 &\leq \|f_n(0)\|_{\dot{H}^{2-2\alpha}}^2 e^{C_\alpha t \varepsilon^{2\alpha/(2\alpha-1)}} \\ &\leq \left\|J_n\theta^0\right\|_{\dot{H}^{2-2\alpha}}^2 e^{C_\alpha \varepsilon^{2\alpha/(2\alpha-1)} \inf(T,T_n)} \\ &\leq \left\|J_n\theta^0\right\|_{\dot{H}^{2-2\alpha}}^2 e^{C_\alpha \varepsilon^{2\alpha/(2\alpha-1)} T}.\end{aligned} \tag{50}$$

For the given value of ε, we have that, for all $t \in [0, \inf(T, T_n))$,

$$\|f_n(t)\|_{\dot{H}^{2-2\alpha}}^2 \leq 2\left\|\theta^0\right\|_{\dot{H}^{2-2\alpha}}^2; \tag{51}$$

thus,

$$\sup_{t\in[0,\inf(T,T_n))} \|f_n(t)\|_{\dot{H}^{2-2\alpha}} \leq \sqrt{2}\left\|\theta^0\right\|_{\dot{H}^{2-2\alpha}} < 2c_\alpha. \tag{52}$$

It follows that $T_n > T$ and, for all $t \in [0, T]$,

$$\|f_n(t)\|_{\dot{H}^{2-2\alpha}}^2 + \int_0^t \|f_n(\tau)\|_{\dot{H}^{2-\alpha}}^2 d\tau \leq 2\left\|\theta^0\right\|_{\dot{H}^{2-2\alpha}}^2, \tag{53}$$

which proves formula (22), and the proof of Proposition 8 is finished.

Now we intend to study the behavior of the solution at infinity. We claim to prove that, for all $s \geq 2 - 2\alpha$,

$$\|\theta(t)\|_{\dot{H}^s} = O\left(\frac{1}{t^{(s-(2-2\alpha))/2\alpha}}\right), \quad t \longrightarrow \infty. \tag{54}$$

We can suppose that $s > 2 - 2\alpha$. We have

$$\begin{aligned}\|\theta(t)\|_{\dot{H}^s}^2 &= \int |\xi|^{2s}\left|\widehat{\theta}(t,\xi)\right|^2 d\xi \\ &= \int |\xi|^{2s-2(2-2\alpha)} e^{-t^{1/2\alpha}|\xi|} |\xi|^{2(2-2\alpha)} e^{t^{1/2\alpha}|\xi|} \left|\widehat{\theta}\right|^2 d\xi \\ &\leq t^{-(2s-2(2-2\alpha))/2\alpha} \\ &\quad \times \int \left(|\xi|\, t^{1/2\alpha} e^{-1/(2s-2(2-2\alpha))t^{1/2\alpha}|\xi|}\right)^{2s-2(2-2\alpha)} \\ &\qquad \times |\xi|^{2(2-2\alpha)} e^{t^{1/2\alpha}|\xi|} \left|\widehat{\theta}\right|^2 d\xi \\ &\leq M t^{-(2s-2(2-2\alpha))/2\alpha} \int |\xi|^{2(2-2\alpha)} e^{t^{1/2\alpha}|\xi|} \left|\widehat{\theta}\right|^2 d\xi,\end{aligned} \tag{55}$$

where $M := \sup_{x>0}(xe^{-(1/(2s-2(2-2\alpha)))x})^{2s-2(2-2\alpha)}$.

Using (22), we get

$$\|\theta(t)\|_{\dot{H}^s}^2 \leq M\left\|\theta^0\right\|_{\dot{H}^{2-2\alpha}}^2 t^{-(s-(2-2\alpha))/\alpha} \tag{56}$$

and the proof of Theorem 3 is finished. □

Remark 9. (i) Combining Theorems 2 and 3, we can obtain, for $2/3 \leq \alpha \leq 1$ and $s \geq 2 - 2\alpha$,

$$\|\theta(t)\|_{\dot{H}^s}^2 = o\left(t^{-(s-(2-2\alpha))/2\alpha}\right), \quad t \longrightarrow \infty. \tag{57}$$

Indeed, from (34), $\|\theta(t)\|_{\dot{H}^{2-2\alpha}} < c_\alpha$, for all $t \geq 0$. For $T > 0$, we consider the following system:

$$\begin{gathered}\partial_t v + (-\Delta)^\alpha v + (V \cdot \nabla) v = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^2, \\ V := (-\Delta)^{-1/2}\left(-\partial_2 v, \partial_1 v\right), \\ v(0,\cdot) = \theta\left(\frac{T}{2},\cdot\right) \quad \text{in } \mathbb{R}^2.\end{gathered} \tag{58}$$

This system has a Leray unique solution v_T that satisfies, for all $t > 0$,

$$\left\|v_T(t,\cdot)\right\|_{\dot{H}^s} \leq \frac{\sqrt{M}\left\|v_T(0,\cdot)\right\|_{\dot{H}^{2-2\alpha}}}{t^{(s-(2-2\alpha))/2\alpha}}. \tag{59}$$

From the uniqueness of the solution, we have $v_T(t,\cdot) = \theta(t + T/2, \cdot)$; then

$$\left\|\theta\left(t + \frac{T}{2},\cdot\right)\right\|_{\dot{H}^s} \leq \frac{\sqrt{M}\|\theta(T/2,\cdot)\|_{\dot{H}^{2-2\alpha}}}{t^{(s-(2-2\alpha))/2\alpha}}. \tag{60}$$

For $t = T/2$, we have

$$\|\theta(T,\cdot)\|_{\dot{H}^s} \leq \frac{2^{(s-(2-2\alpha))/2\alpha}\sqrt{M}\|\theta(T/2,\cdot)\|_{\dot{H}^{2-2\alpha}}}{T^{(s-(2-2\alpha))/2\alpha}}. \tag{61}$$

Combining this inequality with the result of Theorem 2, we obtain the desired result.

(ii) If $\alpha \in (1/2, 2/3)$, we do not know if $\limsup_{t\to\infty}\|\theta(t)\|_{\dot{H}^{(2-2\alpha)}} = 0$ holds. But this result depends on the lower frequencies. Indeed, for $\delta > 0$ and $\varepsilon > 0$, we have

$$\begin{aligned}&\int_{|\xi|>\delta} |\xi|^{2(2-2\alpha)}\left|\widehat{\theta}(t,\xi)\right|^2 d\xi \\ &\quad = \int_{|\xi|>\delta} |\xi|^{-2\varepsilon}|\xi|^{2(2-2\alpha)+2\varepsilon}\left|\widehat{\theta}(t,\xi)\right|^2 d\xi \\ &\quad \leq \delta^{-2\varepsilon}\int_{\mathbb{R}^2} |\xi|^{2(2-2\alpha)+2\varepsilon}\left|\widehat{\theta}(t,\xi)\right|^2 d\xi \\ &\quad \leq \delta^{-2\varepsilon}\|\theta(t)\|_{\dot{H}^{(2-2\alpha)+\varepsilon}}^2.\end{aligned} \tag{62}$$

By Theorem 3, we obtain

$$\begin{aligned}\int_{|\xi|>\delta} |\xi|^{2(2-2\alpha)}\left|\widehat{\theta}(t,\xi)\right|^2 d\xi &\leq M\left\|\theta^0\right\|_{\dot{H}^{(2-2\alpha)}}^2 \delta^{-2\varepsilon} t^{-\varepsilon/\alpha} \\ &\leq M\left\|\theta^0\right\|_{\dot{H}^{(2-2\alpha)}}^2 \left(\delta^{2\alpha}t\right)^{-\varepsilon/\alpha}.\end{aligned} \tag{63}$$

Then, for $\delta = t^{-1/2\alpha+a}$, $a > 0$, we have

$$\begin{gathered}\int_{|\xi|>\delta} |\xi|^{2(2-2\alpha)}\left|\widehat{\theta}(t,\xi)\right|^2 d\xi \leq M\left\|\theta^0\right\|_{\dot{H}^{(2-2\alpha)}}^2 t^{-a\varepsilon/\alpha}, \\ \limsup_{t\to\infty} \int_{|\xi|>t^{-1/2\alpha+a}} |\xi|^{2(2-2\alpha)}\left|\widehat{\theta}(t,\xi)\right|^2 d\xi = 0.\end{gathered} \tag{64}$$

Then, to prove the result,

$$\limsup_{t\to\infty}\|\theta(t)\|_{\dot H^{(2-2\alpha)}} = 0. \tag{65}$$

It suffices to prove that

$$\limsup_{t\to\infty}\int_{|\xi|<t^{-1/2\alpha+a}}|\xi|^{2(2-2\alpha)}\left|\widehat{\theta}(t,\xi)\right|^2 d\xi = 0. \tag{66}$$

4. Proof of Theorem 4

First Step. Using the approximate system (24) and inequality (17),

$$\partial_t\|f_n\|_{L^2}^2 + 2\|f_n\|_{\dot H^\alpha}^2 \le 2\varepsilon\|f_n\|_{\dot H^{1/2}}^2 + C_\alpha\|f_n\|_{\dot H^{2-2\alpha}}\|f_n\|_{\dot H^\alpha}^2. \tag{67}$$

Then,

$$\int_\xi e^{t^{1/2\alpha}|\xi|}\left|\hat\theta(t,\xi)\right|^2 d\xi + \int_0^t\int_\xi e^{\tau t^{-1+1/2\alpha}|\xi|}|\xi|^{2\alpha}\left|\hat\theta(\tau,\xi)\right|^2 d\xi\, d\tau \le 2\left\|\theta^0\right\|_{L^2}^2. \tag{68}$$

Second Step. From relation (22), we deduce that

$$\begin{aligned}\|\theta\|_{\dot H^s}^2 &\le \int_\xi |\xi|^{2s}e^{-t^{1/2\alpha}|\xi|}e^{t^{1/2\alpha}|\xi|}|\mathcal{F}(\theta)|^2 d\xi\\ &\le t^{-s/\alpha}\sup_{x\ge0}x^{2s}e^{-x}\cdot 2\left\|\theta^0\right\|_{L^2}^2\\ &\le Ct^{-s/\alpha}\left\|\theta^0\right\|_{L^2}^2.\end{aligned} \tag{69}$$

Then, the proof is achieved.

Remark 10. If $2/3\le\alpha\le1$ and $s\ge0$, we have

$$\|\theta(t)\|_{\dot H^s}^2 = o\left(t^{-s/2\alpha}\right),\quad t\longrightarrow\infty. \tag{70}$$

5. General Properties of Global Solutions

Theorem 11. *Let θ be a global solution of $(\mathcal{S}_\alpha)$ such that*

$$\theta\in\mathcal{C}\left(\mathbb{R}^+,\dot H^{2-2\alpha}\left(\mathbb{R}^2\right)\right). \tag{71}$$

Then, $\theta\in\mathcal{C}_b(\mathbb{R}^+,\dot H^{2-2\alpha}(\mathbb{R}^2))\cap L^2([T,\infty),\dot H^{2-\alpha}(\mathbb{R}^2))$ for some $T>0$. Moreover, for all $s\ge 2-2\alpha$,

$$\|\theta(t)\|_{\dot H^s} = o\left(t^{-(s-(2-2\alpha))/2\alpha}\right),\quad t\longrightarrow\infty. \tag{72}$$

Combining the energy estimate

$$\|\theta\|_{L^2}^2 + 2\int_0^t\|\theta\|_{\dot H^\alpha}^2 d\tau \le \left\|\theta^0\right\|_{L^2}^2 \tag{73}$$

and the conclusion of Theorems 4 and 11, we get the following.

Theorem 12. *Let θ be a global solution of $(\mathcal{S}_\alpha)$ such that*

$$\theta\in\mathcal{C}\left(\mathbb{R}^+,H^{2-2\alpha}\left(\mathbb{R}^2\right)\right). \tag{74}$$

Then, $\theta\in\mathcal{C}_b(\mathbb{R}^+,H^{2-2\alpha}(\mathbb{R}^2))\cap L^2([T,\infty),\dot H^{2-\alpha}(\mathbb{R}^2))$ for some $T>0$.

Moreover, for all $s\ge0$,

$$\|\theta(t)\|_{\dot H^s} = o\left(t^{-s/2\alpha}\right),\quad t\longrightarrow\infty. \tag{75}$$

Remarks 13. (a) Let $\alpha\in(1/2,1]$ and let θ be a global solution of $(\mathcal{S}_\alpha)$ such that

$$\theta\in\mathcal{C}\left(\mathbb{R}^+,\dot H^{2-2\alpha}\left(\mathbb{R}^2\right)\right). \tag{76}$$

Using the Sobolev injection,

$$L^p\left(\mathbb{R}^2\right)\hookrightarrow\dot H^s\left(\mathbb{R}^2\right),\quad\text{with }\frac{1}{p}+\frac{s}{2}=\frac{1}{2};\ 0<s<1. \tag{77}$$

We conclude that, for all $p\in[2/(1-(2-2\alpha)),\infty)$,

$$\|\theta(t)\|_{L^p} = o\left(t^{-(1-2/p-(2-2\alpha))/2\alpha}\right),\quad t\longrightarrow\infty. \tag{78}$$

(b) Let θ be a global solution of $(\mathcal{S}_\alpha)$ such that

$$\theta\in\mathcal{C}\left(\mathbb{R}^+,H^{2-2\alpha}\left(\mathbb{R}^2\right)\right); \tag{79}$$

then, for all $p\in(2,\infty)$,

$$\|\theta(t)\|_{L^p} = o\left(t^{-(p-2)/2p\alpha}\right),\quad t\longrightarrow\infty. \tag{80}$$

Using the classical interpolation inequality

$$\|f\|_{L^\infty(\mathbb{R}^2)}\le C(s)\|f\|_{L^2(\mathbb{R}^2)}^{1-1/s}\|f\|_{\dot H^s(\mathbb{R}^2)}^{1/s},\quad s\in(1,\infty), \tag{81}$$

and Theorem 12, we get

$$\|\theta(t)\|_{L^\infty} = o\left(t^{-1/2\alpha}\right),\quad t\longrightarrow\infty. \tag{82}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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