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On Wavelet and Leader Wavelet Based Large Deviation Multifractal Formalisms for Non-uniform Hölder Functions

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Abstract Recently, a large deviation multifractal formalism based on histograms of wavelet leader coefficients, compared to some other wavelet-based formalisms, was proved to be efficient for uniform Hölder functions. In this paper, we extend this efficiency for non-uniform Hölder functions. We first obtain optimal bounds for both wavelet and wavelet leader histograms for all functions in the critical Besov space $B_t^{m/t,q}(\mathbb{T})$, where $t, q > 0$ and $\mathbb{T}$ is the unit torus of $\mathbb{R}^m$. We then compute these histograms for quasi-all functions in $B_t^{m/t,q}(\mathbb{T})$, in the sense of Baire Category. Although, increasing parts of these histograms have increasing visibility, they coincide only if $0 < q \leq t$. If moreover $q \leq 1$, then wavelet leader histograms method covers the Hölder spectrum for all $t > 0$, however wavelet histograms method covers it only if $0 < q \leq t$.

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1 Introduction

Multifractal formalisms are formulas derived from global quantities extracted from a signal to compute its Hölder spectrum. The most widespread use the thermodynamic method. For locally bounded functions, Arneodo et al. [1,2] (resp. Jaffard [21,23]) asserted that the Hölder spectrum is given by the Legendre transform of a scaling function based on the wavelet transform (resp. coefficients) of f. This assertion is reminiscent to a conjecture of Frisch and Parisi [19] for turbulence. Mandelbrot [29–32] had associated fractals to measures (or functions) by introducing multiplicative cascades for the dissipation of energy in turbulent flows. The validity of the thermodynamic method has been studied under self-similarity assumptions on f [1,7–12,16,21], or for a class of particular random processes [22], or even for specific functions f [13,20]. It was also proved in a Baire (resp. prevalence) generic setting by Jaffard [23] (resp. Fraysse [18]) in the Besov space $B_t^{s,q}(\mathbb{R}^m)$ for $s > m/t$.

Since the Legendre transform is always concave, then functions with non concave spectra are counter-examples for the thermodynamic method. Large deviation formalisms based on statistics of histograms of wavelet coefficients and wavelet leaders were proposed as alternative methods.

The wavelet density method due to Aubry and Jaffard [4], asserts that for a uniform Hölder function f, the Hölder spectrum d_f is given by the wavelet density ρ_f, which in some way gives the asymptotic behavior of the number $2^{j\rho_f(\alpha)}$ of wavelet coefficients that have $2^{-\alpha j}$ magnitude. In [14], Ben Slimane proved this method Baire generically in $B_t^{s,q}(\mathbb{T})$ for $s > m/t$, where $\mathbb{T}$ is the unit torus of $\mathbb{R}^m$.

Since ρ_f may depend on the chosen wavelet basis, Jaffard [24] replaced it by its increasing hull, the so called wavelet profile ν_f. The transformation of ν_f to its increasing-visibility function is called the sunny wavelet profile d_f^{ν}. The sunny wavelet profile method asserts that $d_f = d_f^{\nu}$ on the part where d_f has increasing visibility (i.e., $d_f(\alpha)/\alpha$ is increasing). For uniform Hölder functions, Aubry and Jaffard [4] proved that d_f^{ν} yields an upper bound for d_f. They also proved that equality holds for some random wavelet series [4] and specific functions [24]. The validity was also studied in a generic setting by Aubry, Bastin and Dispa [3,5].

In order to cover non concave Hölder spectra not limited to the increasing part, Bastin et al. [6] suggested the wavelet leaders profile $\tilde{d}_f$. Roughly speaking, $\tilde{d}_f$ quantifies at large scales j the asymptotic number $2^{j\tilde{d}_f^{+}(\alpha)}$ (resp. $2^{j\tilde{d}_f^{-}(\alpha)}$) of wavelet leaders of size larger than $2^{-\alpha j}$ (resp. smaller than $2^{-\alpha j}$). The main tool is the Jaffard characterization [25] of the Hölder exponent of any uniform Hölder function by decay conditions of the wavelet leaders in the cone of influence. It is proved that, for uniform Hölder functions, $\tilde{d}_f$ yields an upper bound for the Hölder spectrum. It is also proved that equality holds for some classical models used in signal (turbulence, finance) and image processing. In [17], Esser et al. implemented the wavelet leader profile method. They also proved its numerical efficiency compared to many formulas (above and others) for the fractional Brownian motion, Lévy process, Lévy process with a Brownian part, and multiplicative cascades. Many theoretical results were also obtained, among them, it was shown that, if f is uniform Hölder, then $\tilde{d}_f \leq d_f^\nu$, moreover on the increasing part of $\tilde{d}_f$ the equality $\tilde{d}_f = d_f^\nu$ holds if and only if $\tilde{d}_f$ has increasing visibility.

In this paper, we aim to study the validity of these assertions for non-uniform Hölder functions. For these functions, the upper bound of the Hölder exponent by the above decay conditions of the wavelet leaders in the cone of influence remains true. But, in [27], Jaffard and Meyer proved that a weaker lower bound is also possible under the sole assumption that f belongs to the critical Besov space $B_t^{m/t,q}(\mathbb{R}^m)$ when $t > 0$ and $0 < q < 1$. The smaller q is, the more this lower bound is close to the upper bound. This allowed them to bound (resp. compute) the Hölder spectrum for all (resp. quasi-all (in the sense of Baire)) functions f in $B_t^{m/t,q}(\mathbb{R}^m)$ when $t > 0$ and $0 < q \leq 1$. This was the starting point of this paper. We first obtain general upper bounds for d_f^ν and the increasing part $\tilde{d}_f^+$ of $\tilde{d}_f$ for all functions in $B_t^{m/t,q}(\mathbb{T})$, for all $t, q > 0$. We then compute, for all $t, q > 0$, both d_f^ν and $\tilde{d}_f^+$ for quasi-all functions in $B_t^{m/t,q}(\mathbb{T})$. We deduce the optimality of the obtained upper bounds. We also show that the increasing part of $\tilde{d}_f$ has increasing visibility, on which, $\tilde{d}_f = d_f^\nu$ if $0 < q \leq t$, while $\tilde{d}_f > d_f^\nu$ if $0 < t < q$. If moreover $q \leq 1$, then the increasing part of d_f coincides with that of $\tilde{d}_f$ (therefore it doesn't coincide with that of d_f^ν for $0 < t < q$). The decreasing part is reduced to a single point $-\infty$. This result also confirms the effectiveness of the wavelet leader profile method compared to the sunny wavelet profile method.

Note that if $s > m/t$ then all functions in $B_t^{s,q}(\mathbb{T})$ are uniform Hölder, and quasi-all functions in this space satisfy the thermodynamic formalism (see [23]) on the increasing part of the Hölder spectrum, on which the wavelet profile ν_f has increasing visibility (see [14]). It follows that quasi-all functions in this space satisfy both wavelet leader profile and sunny wavelet profile formalisms.

Let us now describe in details these formalisms and state our main results.

2 Multifractal Formalisms and Main Results

We first recall the sunny wavelet and leader profile methods. For more details see [6,17] and references therein. We consider functions on the unit torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (i.e 1-periodic functions). Extensions to higher dimension are straightforward.

Let $x \in [0, 1]$ and $\alpha > 0$ non integer. Recall that $f \in C^\alpha(x)$ if there exist a polynomial P of degree less than α and a constant C such that, in a neighborhood of x

$$|f(y) - P(y - x)| \leq C|y - x|^\alpha. \quad (1)$$

The Hölder exponent of f at x is defined as

$$h_f(x) = \sup\left\{\alpha \ : \ f \in C^\alpha(x)\right\} . \quad (2)$$

The Hölder spectrum of f is the function $d_f(\alpha)$ defined for each $\alpha > 0$ as the Hausdorff dimension of the set $E_f(\alpha)$ of points x such that $h_f(x) = \alpha$. Conventionally $dim\emptyset = -\infty$. In many situations (multifractal analysis, PDEs,...) one wishes to compute or bound this spectrum or its increasing hull $D_f(\alpha) = \sup_{\alpha' \leq \alpha} d_f(\alpha)$. Note that $D_f(\alpha)$ coincides with the upper Hölder spectrum of f, which is the Hausdorff dimension of the set E_f^α of points x such that $h_f(x) \leq \alpha$.

Let ψ be a mother wavelet in the Schwartz class such that the constant function 1, together with the periodized functions $2^{j/2}\psi_{j,k}(x) := 2^{j/2}\sum_{l\in\mathbb{Z}} \psi(2^j(x-l)-k)$, $j \geq 0$, $k \in \{0, \ldots, 2^j - 1\}$, form an orthonormal basis of the space $L^2(\mathbb{T})$ (see [28]). Denote

$$C_{j,k} = 2^j \int_0^1 f(x)\psi_{j,k}(x)\, dx \tag{3}$$

the wavelet coefficient of a function f in $L^2(\mathbb{T})$ at scale j and position k (with the usual modification when f is a tempered distribution periodic over $\mathbb{Z}$). Let λ denote the interval $[k2^{-j}, k2^{-j} + 2^{-j})$. Write C_λ (resp. ψ_λ) instead of $C_{j,k}$ (resp. $\psi_{j,k}$). If j is fixed, denote by Λ_j the set of all intervals λ where $k \in \{0, \ldots, 2^j - 1\}$.

Remark 1 In this paper, all functions are in $C^0(\mathbb{T})$, i.e,

$$\exists\, C > 0 \qquad \forall\, \lambda \qquad |C_\lambda| \leq C\,. \tag{4}$$

Let $\sharp$ mean cardinality. Let $\alpha \geq 0$. For each $j \geq 0$ consider

$$N_j(\alpha) = \sharp\{\lambda \in \Lambda_j \;:\; |C_\lambda| \geq 2^{-\alpha j}\} \tag{5}$$

The wavelet profile ν_f is defined by

$$\nu_f(\alpha) = \lim_{\varepsilon\mapsto 0} \limsup_{j\mapsto\infty} \frac{\log N_j(\alpha+\varepsilon)}{\log(2^j)} \in \{-\infty\} \cup [0,1]\,. \tag{6}$$

The wavelet density ρ_f is defined by

$$\rho_f(\alpha) = \inf_{\varepsilon>0} \limsup_{j\mapsto\infty} \frac{\log(N_j(\alpha+\varepsilon) - N_j(\alpha-\varepsilon))}{\log(2^j)} \in \{-\infty\} \cup [0,1]\,. \tag{7}$$

Heuristically, this means that at large scale j there are about $2^{\nu_f(\alpha)j}$ (resp. $2^{\rho_f(\alpha)j}$) wavelet coefficients of size larger than (resp. of order) $2^{-\alpha j}$.

The function ν_f is increasing and right-continuous. It is also the increasing hull of ρ_f.

In [24], it is proved that it does not depend on the chosen wavelet basis. The function ρ_f is upper-semi-continuous but may depend on the chosen wavelet basis.

Clearly, since $f \in C^0(\mathbb{T})$, we can extend ν_f and ρ_f to negative values of α, by putting $\nu_f(\alpha) = -\infty$ and $\rho_f(\alpha) = -\infty$.

Let

$$\alpha_{min} = \inf\{\alpha \geq 0 \ : \ \nu_f(\alpha) \geq 0\}$$

and

$$\alpha_{max} = \inf_{\alpha \geq \alpha_{min}} \frac{\alpha}{\nu_f(\alpha)} \ .$$

The sunny wavelet profile is given by

$$d^{\nu}(\alpha) = \begin{cases} -\infty & \text{if } \alpha < \alpha_{min} \\ \alpha \sup\limits_{\alpha' \in (0,\alpha]} \dfrac{\nu_f(\alpha')}{\alpha'} & \text{if } \alpha_{min} \leq \alpha \leq \alpha_{max} \\ 1 & \text{if } \alpha > \alpha_{max} \ . \end{cases} \tag{8}$$

Definition 1 Let $0 \leq a < b \leq \infty$. A positive function g has increasing visibility on $[a, b]$ if the function $g(x)/x$ is increasing on $(a, b]$.

The sunny wavelet profile has the property of increasing visibility on $[\alpha_{min}, \alpha_{max}]$. The sunny wavelet profile method due to Aubry and Jaffard [4] asserts that, if d_f has increasing visibility on an interval $[a, b]$ then

$$\forall \ \alpha \in (a, b] \qquad d_f(\alpha) = d_f^{\nu}(\alpha) \ . \tag{9}$$

In [4], it is proved that it yields an upper bound for uniform Hölder functions

$$\forall \ \alpha \qquad d_f(\alpha) \leq d^{\nu}(\alpha) \ . \tag{10}$$

Recall that f is uniform Hölder if there exists $\alpha > 0$ such that $f \in C^{\alpha}(\mathbb{T})$, in the sense that, (1) holds for any x and y in $[0, 1]$ and C is uniform. This implies that (see [33])

$$\exists \ C > 0 \qquad \forall \ \lambda \qquad |C_\lambda| \leq C 2^{-\alpha j} \ . \tag{11}$$

The wavelet leader associated to a dyadic interval λ is defined by

$$e_\lambda = \sup_{\lambda' \subset \lambda} |C_{\lambda'}| \tag{12}$$

where the supremum is over all dyadic intervals $\lambda' = [k' 2^{-j'}, k' 2^{-j'} + 2^{-j'})$ included in λ. Thanks to (11), this supremum is finite.
Let $\alpha \geq 0$. For each $j \geq 0$ consider

$$M_j^+(\alpha) = \sharp\{\lambda \in \Lambda_j \ : \ e_\lambda \geq 2^{-\alpha j}\} \tag{13}$$

and

$$M_j^-(\alpha) = \sharp\{\lambda \in \Lambda_j \ : \ e_\lambda \leq 2^{-\alpha j}\} \, . \tag{14}$$

The increasing leader profile $\tilde{d}_f^+$ is defined for $\alpha \in [0, \infty]$ by

$$\tilde{d}_f^+(\alpha) = \lim_{\varepsilon \mapsto 0} \limsup_{j \mapsto \infty} \frac{\log M_j^+(\alpha + \varepsilon)}{\log(2^j)} \in \{-\infty\} \cup [0, 1] \, . \tag{15}$$

The decreasing leader profile $\tilde{d}_f^-$ is defined for $\alpha \in [0, \infty)$ by

$$\tilde{d}_f^-(\alpha) = \lim_{\varepsilon \mapsto 0} \limsup_{j \mapsto \infty} \frac{\log M_j^-(\alpha - \varepsilon)}{\log(2^j)} \in \{-\infty\} \cup [0, 1] \, . \tag{16}$$

Heuristically, this means that at large scale j there are about $2^{\tilde{d}_f^+(\alpha) j}$ (resp. $2^{\tilde{d}_f^-(\alpha) j}$) wavelet leaders of size larger (resp. smaller) than $2^{-\alpha j}$.

The increasing leader profile is increasing, right-continuous, and $\tilde{d}_f^+(\infty) = 1$.

The decreasing leader profile is decreasing, left-continuous, and $\tilde{d}_f^-(0) = 1$.

Clearly, since $f \in C^0(\mathbb{T})$, we can extend $\tilde{d}_f^+$ and $\tilde{d}_f^-$ to negative values of α, by putting $\tilde{d}_f^+(\alpha) = -\infty$ and $\tilde{d}_f^-(\alpha) = 1$.

In [6], it is proved that, if f is uniform Hölder, then $\tilde{d}_f^+$ and $\tilde{d}_f^-$ do not depend on the chosen wavelet function ψ in the Schwartz class. Let

$$\alpha_s = \inf\{\alpha \in [0, \infty] \ : \ \tilde{d}_f^+(\alpha) = 1\} \, . \tag{17}$$

The leader profile function is given by

$$\tilde{d}_f(\alpha) = \begin{cases} \tilde{d}_f^+(\alpha) \text{ if } \alpha < \alpha_s \\ \tilde{d}_f^-(\alpha) \text{ if } \alpha \geq \alpha_s \, . \end{cases} \tag{18}$$

The leaders profile method [6] asserts that

$$\forall \, \alpha \qquad d_f(\alpha) = \tilde{d}_f(\alpha) \, . \tag{19}$$

In [6], it is shown that the leaders profile method yields an upper bound for uniform Hölder functions

$$\forall \, \alpha \qquad d_f(\alpha) \leq \tilde{d}_f(\alpha) \, . \tag{20}$$

In [17], it is proved that, if f is uniform Hölder, then

$$\forall \, \alpha \qquad \tilde{d}_f(\alpha) \leq d^\nu(\alpha) \, . \tag{21}$$

It is also shown that, if f is uniform Hölder, then on $[\alpha_{min}, \alpha_s]$

$$\tilde{d}_f = d^\nu \iff \tilde{d}_f \text{ is with increasing visibility } . \tag{22}$$

Remark 2 Since, only wavelet leaders for $j \geq 0$ are needed in the values of Hölder exponent and increasing and decreasing leader profiles, then from now on, we will identify functions that have the same wavelet coefficients C_λ. With this identification, Besov spaces $B_p^{s,q}(\mathbb{T})$, for $0 \leq s \leq \infty, 0 < p \leq \infty, 0 < q \leq \infty$, are characterized by (see [15,33])

$$f \in B_p^{s,q}(\mathbb{T}) \iff \left(\sum_{j\geq 0} \left(\sum_{\lambda \in \Lambda_j} |C_\lambda 2^{(s-\frac{1}{p})j}|^p \right)^{q/p} \right)^{1/q} < \infty \tag{23}$$

(with the usual modification when $p = \infty$ and/or $q = \infty$).

Besov spaces are Baire spaces (see [34]). Any countable intersection of dense open sets is dense and called a generic set. If a given property (P) in $B_p^{s,q}(\mathbb{T})$ holds in a generic set, then we say that quasi-all functions in $B_p^{s,q}(\mathbb{T})$ satisfy (P).

We are now in a position to state our main results. We will first obtain general optimal upper bounds for the sunny wavelet and increasing leader profiles for functions in $B_t^{1/t,q}(\mathbb{T})$. For the latter, we will separate cases $q \leq t$ and $t < q$.

Theorem 1 1. (a) *For all* $f \in B_t^{\frac{1}{t},q}(\mathbb{T})$

$$d_f^\nu(\alpha) \begin{cases} = -\infty & if\ \alpha < 0 \\ \leq \alpha t & if\ 0 \leq \alpha \leq 1/t \\ \leq 1 & if\ \alpha > 1/t\ . \end{cases} \tag{24}$$

(b) *If* $0 < q \leq t$, *then for all* $f \in B_t^{\frac{1}{t},q}(\mathbb{T})$

$$\tilde{d}_f^+(\alpha) \begin{cases} = -\infty & if\ \alpha < 0 \\ \leq \alpha t & if\ 0 \leq \alpha \leq 1/t \\ \leq 1 & if\ \alpha > 1/t\ . \end{cases} \tag{25}$$

(c) *If* $0 < t < q$, *then for all* $f \in B_t^{\frac{1}{t},q}(\mathbb{T})$

$$\tilde{d}_f^+(\alpha) \begin{cases} = -\infty & if\ \alpha < 0 \\ \leq \alpha q & if\ 0 \leq \alpha \leq 1/q \\ \leq 1 & if\ \alpha > 1/q\ . \end{cases} \tag{26}$$

2. *All above upper bounds are optimal, namely in each case there exists a function F in the corresponding space for which the above upper bounds are equalities.*

In the next section, we will prove Theorem 1.

In the fourth Section, we will first prove that actually optimality in Theorem 1 holds Baire generically.

Theorem 2 1. (a) *Quasi-all functions in* $B_t^{1/t,q}(\mathbb{T})$ *satisfy*

$$d_f^\nu(\alpha) = \begin{cases} -\infty & if\ \alpha < 0 \\ \alpha t & if\ 0 \le \alpha \le 1/t \\ 1 & if\ \alpha > 1/t\ . \end{cases} \tag{27}$$

(b) *If* $0 < q \le t$, *then quasi-all functions in* $B_t^{1/t,q}(\mathbb{T})$ *satisfy*

$$\tilde{d}_f^+(\alpha) = \begin{cases} -\infty & if\ \alpha < 0 \\ \alpha t & if\ 0 \le \alpha \le 1/t \\ 1 & if\ \alpha > 1/t\ . \end{cases} \tag{28}$$

(c) *If* $0 < t < q$, *then quasi-all functions in* $B_t^{1/t,q}(\mathbb{T})$ *satisfy*

$$\tilde{d}_f^+(\alpha) = \begin{cases} -\infty & if\ \alpha < 0 \\ \alpha q & if\ 0 \le \alpha \le 1/q \\ 1 & if\ \alpha > 1/q\ . \end{cases} \tag{29}$$

2. *The generic set* $A_{q,t}$ *of* (27) *is the same as the one of* (28)*, but is different from the one of* (29) *denoted* A_t^q.

We therefore deduce the following corollary (see first point) which shows that result (22) can not be always extended for non-uniform Hölder functions.

Corollary 1

- *If* $0 < t < q$ *then for all* $f \in A_t^q$ *the increasing part of* $\tilde{d}_f$ *corresponds to* $\alpha \in [0, 1/q]$*, on which it has increasing visibility, and*

$$\forall\, \alpha \in (0, 1/q] \quad \tilde{d}_f(\alpha) > d_f^\nu(\alpha)\ .$$

- *If* $0 < q \le t$ *then for all* $f \in A_{q,t}$ *the increasing part of* $\tilde{d}_f$ *corresponds to* $\alpha \in [0, 1/t]$*, on which it has increasing visibility, and*

$$\forall\, \alpha \in [0, 1/t] \quad \tilde{d}_f(\alpha) = d_f^\nu(\alpha)\ .$$

Actually, the generic set $A_{q,t}$ (resp. A_q^t) is the same as the one (resp. is a correctly reduced generic set of the one) of Jaffard and Meyer [27], in which they computed the Hölder spectrum for functions in $B_t^{m/t,q}(\mathbb{R}^m)$ when $t > 0$ and $0 < q \le 1$. (Note that, for $q > 1$, it was proved (see [27]) that quasi-all functions in $B_t^{m/t,q}(\mathbb{R}^m)$ are not locally bounded, so Hölder spectra are meaningless). This leads to the following theorem which confirms the effectiveness of the wavelet leader profile method compared to the sunny wavelet profile method.

Theorem 3

- *If* $0 < t < q \leq 1$ *then for all* $f \in A_t^q$ *the increasing part of* d_f *corresponds to* $\alpha \in [0, 1/q]$*, on which it has increasing visibility, and*

$$\forall \alpha \in (0, 1/q] \quad d_f(\alpha) = \tilde{d}_f(\alpha) > d_f^{\nu}(\alpha) \ .$$

The decreasing part is reduced to a single point $-\infty$*.*

- *If* $0 < q \leq t$ *and* $q \leq 1$*, then for all* $f \in A_{q,t}$ *the increasing part of* d_f *corresponds to* $\alpha \in [0, 1/t]$*, on which it has increasing visibility, and*

$$\forall \alpha \in [0, 1/t] \quad d_f(\alpha) = \tilde{d}_f(\alpha) = d_f^{\nu}(\alpha) \ .$$

The decreasing part is reduced to a single point $-\infty$*.*

3 Proof of Theorem 1

3.1 Proof of the First Point in Theorem 1

(a) If $f \in B_t^{1/t,q}(\mathbb{T})$ then $f \in B_t^{1/t,\infty}(\mathbb{T})$, i.e.,

$$\sup_{j \geq 0} \left(\sum_{\lambda \in \Lambda_j} |C_\lambda|^t \right)^{1/t} < \infty \ .$$

Fix $\alpha \geq 0$ and $\delta > 0$. For any $\varepsilon > 0$ there exists a sequence (j_n) with $\lim_{n \to \infty} j_n = \infty$ such that

$$N_{j_n}(\alpha + \varepsilon) \geq 2^{j_n(\nu_f(\alpha) - \delta)} \ .$$

Therefore

$$\sum_{\lambda \in \Lambda_{j_n}} |C_\lambda|^t \geq 2^{j_n(\nu_f(\alpha)-\delta)} 2^{-(\alpha+\varepsilon) j_n t} \ .$$

It follows that

$$\nu_f(\alpha) - \delta - (\alpha + \varepsilon)t \leq 0$$

Letting δ and ε tend to 0 we obtain

$$\nu_f(\alpha) \leq \alpha t \tag{30}$$

which yields (24).

(b) We will apply the following proposition.

Proposition 1 *For $p > 0$ define the oscillation space $O_p(\mathbb{T})$ by*

$$f \in O_p(\mathbb{T}) \Longleftrightarrow |f|_p := \sup_{j \geq 0} \left(\sum_{\lambda \in \Lambda_j} e_\lambda^p \right)^{1/p} < \infty \, . \tag{31}$$

- *The following embeddings hold.*

$$\forall \; p > 0 \;\; \forall \; s > 0 \qquad O_p(\mathbb{T}) \hookrightarrow C^0(\mathbb{T}) \, . \tag{32}$$

$$\forall \; 0 < p < p' \qquad O_p(\mathbb{T}) \hookrightarrow O_{p'}(\mathbb{T}) \, . \tag{33}$$

$$\forall \; p > 0 \quad B_p^{\frac{1}{p},p}(\mathbb{T}) \hookrightarrow O_p(\mathbb{T}) \, . \tag{34}$$

- *If $f \in O_p(\mathbb{T})$ for $p > 0$ then*

$$\forall \alpha \geq 0 \quad \tilde{d}_f^+(\alpha) \leq \alpha p \, . \tag{35}$$

Let $0 < q \leq t$. Let $f \in B_t^{1/t,q}(\mathbb{T})$. Since $q \leq t$, then $f \in B_t^{1/t,t}(\mathbb{T})$. But (34) implies that $f \in O_t(\mathbb{T})$. Result (35) yields (25).

(c) Let $0 < t < q$ and $f \in B_t^{1/t,q}(\mathbb{T})$. It follows from embedding (34) that $f \in O_t(\mathbb{T})$. But (33) implies that $f \in O_q(\mathbb{T})$. Result (35) yields (26).

Proof of Proposition 1.

- The space $O_p(\mathbb{T})$ is a particular case of general oscillation spaces $O_p^{s,s'}$ taken in [26]. The proof can be directly deduced from Proposition 2 in [26]. But since our particular case is simple, we will give the proof. The first embedding follows from the fact that

$$\forall \; \lambda \quad |C_\lambda| \leq e_\lambda \leq |f|_p \, .$$

The first embedding implies that

$$\forall \; \lambda \quad e_\lambda \leq |f|_p \, .$$

The second embedding comes from the fact that

$$\sum_{\lambda \in \Lambda_j} e_\lambda^{p'} \leq |f|_p^{p'-p} \sum_{\lambda \in \Lambda_j} e_\lambda^p \, .$$

The third embedding is deduced from the fact that

$$\sum_{\lambda \in \Lambda_j} e_\lambda^p \leq \sum_{\lambda \in \Lambda_j} \sum_{j' \geq j} \sum_{\lambda' \subset \lambda} |C_{\lambda'}|^p = \sum_{j' \geq j} \sum_{\lambda' \in \Lambda_{j'}} |C_{\lambda'}|^p \, .$$

- The proof is similar to the one of (30).

$\square$

3.2 Proof of Optimality in Theorem 1

We will give the proof of each case separately.

3.2.1 Optimality of (24)

Let $j \geq 1$ and $k \in \{0, \ldots, 2^j - 1\}$ be given. Write

$$\frac{k}{2^j} = \frac{K}{2^J} \qquad \text{with } K \text{ odd and } J \leq j. \tag{36}$$

Let

$$a = \frac{2}{t} + \frac{2}{q} + 1. \tag{37}$$

Let

$$F(x) = \sum_{j\geq 1} \sum_{\lambda \in \Lambda_j} \frac{1}{j^a} 2^{-\frac{J}{t}} \, \psi_\lambda(x) \tag{38}$$

Proposition 2 *If F is the function given in (38) then*

$$\nu_F(\alpha) = \begin{cases} -\infty & if\ \alpha < 0 \\ \alpha t & if\ 0 \leq \alpha \leq \frac{1}{t} \\ 1 & if\ \alpha > \frac{1}{t}\ . \end{cases} \tag{39}$$

Proof We will use the following trivial lemma. □

Lemma 1 *For each $1 \leq J \leq j$ there are $2^J/2$ values of k satisfying (36).*

Clearly, ν_f doesn't change if we replace C_λ in (13) by $j^a C_\lambda$. We have

$$\forall\ \lambda \in \Lambda_j \qquad j^a C_\lambda \in [2^{-j/t}, 1]\ .$$

So, it suffices to prove (39) for $\alpha \in [0, 1/t)$.
Let $\lambda \in \Lambda_j$. Relation

$$j^a C_\lambda \geq 2^{-j(\alpha+\varepsilon)} \tag{40}$$

is equivalent to $\frac{J}{t} \leq (\alpha + \varepsilon) j$ and $J \leq j$. This means that

$$J \leq (\alpha + \varepsilon) t j \quad \text{and} \quad J \leq j\ . \tag{41}$$

For ε small enough

$$(\alpha + \varepsilon) t \leq 1\ . \tag{42}$$

Lemma 1 yields

$$\sharp\{\lambda \in \Lambda_j \ : \ j^a C_\lambda \geq 2^{-(\alpha+\varepsilon)j}\} = \frac{1}{2} \sum_{J \leq (\alpha+\varepsilon)tj} 2^J \ .$$

It follows from (41) and (42) that there exists a constant C such that

$$\frac{1}{C} 2^{(\alpha+\varepsilon)j} \leq \sharp\{\lambda \in \Lambda_j \ : \ j^a C_\lambda \geq 2^{-(\alpha+\varepsilon)j}\} \leq C 2^{(\alpha+\varepsilon)tj} \ .$$

Consequently $\nu_F(\alpha) = \alpha t$.
Clearly $\alpha_{min} = 0$, $\alpha_{max} = 1/t$ and

$$\forall \ \alpha \qquad d_F^\nu(\alpha) = \nu_F(\alpha) \ . \tag{43}$$

3.2.2 Optimality of (25) when $0 < q \leq t$

Assume that $0 < q \leq t$. We take the function F given in (38). If $\lambda' \subset \lambda$ then $j' \geq j$ and $J' \geq J$. It follows that

$$\forall \ \lambda \qquad e_\lambda = C_\lambda \ . \tag{44}$$

It follows that

$$\tilde{d}_F^+ = \nu_F \ . \tag{45}$$

It is given by the right-hand term in (39).

3.2.3 Optimality of (26) when $0 < t < q$

Consider the function

$$F(x) = \sum_{j \geq 2} \frac{1}{(j(\log j)^2)^{\frac{1}{q}}} \ \ \psi_{j,k_j}(x) \ , \tag{46}$$

where

$$k_j = (j - 2^r)2^{j-r} \qquad \text{if} \qquad 2^r \leq j < 2^{r+1} \text{ with } r \in \mathbb{N} \ . \tag{47}$$

Proposition 3 *Let F be the function given in (46) and (47). Let $j \geq 2$.*

1. *If $k = k_j$ then*

$$e_\lambda = \frac{1}{(j(\ln j)^2)^{\frac{1}{q}}} \ . \tag{48}$$

2. *If* $k \neq k_j$. *Let* J *be as in* (36).
   (a) *If* $2^J \leq j$ *then there exists* $C > 0$ *independent of* j *and* k *such that*

$$\frac{1/C}{(j(\ln j)^2)^{\frac{1}{q}}} \leq e_\lambda \leq \frac{C}{(j(\ln j)^2)^{\frac{1}{q}}} . \tag{49}$$

   (b) *If* $j < 2^J$ *then there exists* $C > 0$ *independent of* j *and* k *such that*

$$\frac{1/C}{(J^2 2^J)^{\frac{1}{q}}} \leq e_\lambda \leq \frac{C}{(J^2 2^J)^{\frac{1}{q}}} . \tag{50}$$

Proof

1. If $k = k_j$, then $C_\lambda = \frac{1}{(j(\ln j)^2)^{\frac{1}{q}}}$. For all $j' \geq j$, we have $\frac{1}{(j'(\ln j')^2)^{\frac{1}{q}}} \leq C_\lambda$. Thus (48) holds.
2. Let now $k \neq k_j$.
   (a) If $2^J \leq j$ then $J \leq r < j$. Let $j' > j$ with $2^r \leq j' < 2^{r+1}$. Let $\lambda' \in \Lambda_{j'}$ with $\lambda' \subset \lambda$. This implies that $C_{\lambda'} \neq 0$ if and only if $\frac{k'}{2^{j'}} = \frac{j'-2^r}{2^r} = \frac{k}{2^j}$. It follows that $j' > j$ if and only if $j' = 2^r + 2^{r-J}K > j$.
      i. If $2^r \leq j < 2^r + 2^{r-J}K$, then it follows from above that (49) holds.
      ii. If $2^r + 2^{r-J}K \leq j < 2^{r+1}$, take $j' \in [2^{r+1}, 2^{r+2})$ such that $\frac{k'}{2^{j'}} = \frac{j'-2^{r+1}}{2^{r+1}} = \frac{k}{2^j}$. And (49) holds.
   (b) If $j < 2^J$ then $r < J \leq j$. If $2^{r'} \leq j' < 2^{r'+1}$, $\lambda' \in \Lambda_{j'}$, $\lambda' \subset \lambda$ and $C_{\lambda'} \neq 0$ then $\frac{k'}{2^{j'}} = \frac{j'-2^{r'}}{2^{r'}} = \frac{k}{2^j}$. It follows that $r' \geq J$. By taking $r' = J$, we get (50).
□

Proposition 4 *Let* F *be the function given in* (46) *and* (47). *Then*

$$\tilde{d}_F^+(\alpha) = \begin{cases} -\infty & if\ \alpha < 0 \\ \alpha q & if\ 0 \leq \alpha \leq 1/q \\ 1 & if\ \alpha > 1/q\ . \end{cases} . \tag{51}$$

Proof Let $\alpha \geq 0$ and $j \geq 2$ fixed. By (48) and (49), intervals $\lambda \in \Lambda_j$ with $k = k_j$ or ($k \neq k_j$ and $2^J \leq j$), give a contribution $1 + j/2$ to $M_j(\alpha + \varepsilon)$, so a zero contribution in $\tilde{d}_F^+(\alpha)$.
On the other hand, thanks to Lemma 1, by (50), intervals $\lambda \in \Lambda_j$ with $j < 2^J, k \neq k_j$ and $\frac{1/C}{(J^2 2^J)^{\frac{1}{q}}} \leq 2^{-(\alpha+\varepsilon)j}$, give a contribution of the order of $2^{q(\alpha+\varepsilon)j}$ to $M_j(\alpha+\varepsilon)$, if and only if $J \leq q(\alpha+\varepsilon)j$ so a $q\alpha$ contribution in $\tilde{d}_F^+(\alpha)$, if and only if $0 \leq \alpha \leq 1/q$ (because $J \leq j$). □

4 Proof of Theorem 2

4.1 Proof of (27)

Let F be the function given in (38). Since $0 < t < \infty$ and $0 < q < \infty$ then $B_t^{1/t,q}(\mathbb{T})$ is separable. Let (f_n) be a dense sequence in $B_t^{1/t,q}(\mathbb{T})$. Set

$$g_n(x) = \sum_{j<n} \sum_{\lambda\in\Lambda_j} C_\lambda(f_n)\psi_\lambda(x) + \sum_{j\geq n} \sum_{\lambda\in\Lambda_j} C_\lambda(F)\psi_\lambda(x) \; . \tag{52}$$

The generic set is

$$A_{q,t} = \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} B(g_n, r_n) \tag{53}$$

where $B(g_n, r_n)$ is the open ball of $B_t^{1/t,q}(\mathbb{T})$ centered at g_n and radius $r_n = \dfrac{1}{2n^a} 2^{-n/t}$. If $f \in A_{q,t}$, then for infinitely many scales n, we have

$$\forall\, \lambda \in \Lambda_n \qquad |C_\lambda(f)| \geq \frac{1}{2} C_\lambda(F) \; . \tag{54}$$

It follows that

$$\forall\, \alpha \quad d_f^\nu(\alpha) \geq \nu_f(\alpha) \geq \nu_F(\alpha) = d_F^\nu(\alpha) \; .$$

This result together with both (24), Proposition 2 and (43) yield (27).

4.2 Proof of (28)

Assume that $0 < q \leq t < \infty$. Let $A_{q,t}$ be the generic set given in (53). Relation (54) implies that infinitely many scales n, we have

$$\forall\, \lambda \in \Lambda_n \qquad e_\lambda(f) \geq \frac{1}{2} C_\lambda(F) \; . \tag{55}$$

It follows from (44) that

$$\forall\, \lambda \in \Lambda_n \qquad e_\lambda(f) \geq \frac{1}{2} e_\lambda(F) \; . \tag{56}$$

Therefore

$$\forall\, \alpha \quad \tilde{d}_f^+(\alpha) \geq \tilde{d}_F^+(\alpha) \; .$$

This result together with both (24), (45) and Proposition 2 yield (28).

4.3 Proof of (29)

Assume that $0 < t < q < \infty$. The generic set A_t^q is as in (53) with

$$r_n = \frac{1}{2}\frac{1}{(2^{n+1}((n+1)\log 2)^2)^{\frac{1}{q}}} \ . \tag{57}$$

and g_n as in (52) associated to the function F given in (46).
Clearly, from the proof of Proposition 3, wavelet leaders of F are attained, i.e.,

$$\forall\, \lambda \quad \exists\, \lambda' \subset \lambda \quad e_\lambda = C_{\lambda'} \ . \tag{58}$$

Proposition 3 also implies that

$$\forall\, \lambda \in \Lambda_j \qquad e_\lambda(F) \geq \frac{1}{(2^{j+1}((j+1)\log 2)^2)^{\frac{1}{q}}} \ . \tag{59}$$

Proposition 5 *If $f \in A_t^q$, then for infinitely many n's*

$$\forall \lambda \in \Lambda_n \qquad e_\lambda(f) \geq \frac{1}{2} e_\lambda(F) \ . \tag{60}$$

Proof Let $f \in A_t^q$. For infinitely many n's

$$\| f - g_n \| < r_n \ .$$

So

$$\forall \lambda' \quad |C_{\lambda'}(f) - C_{\lambda'}(g_n)| < r_n \ .$$

This implies that

$$\forall\, j' \geq n \;\; \forall\, \lambda' \in \Lambda_{j'} \qquad |C_{\lambda'}(f)| \geq |C_{\lambda'}(g_n)| - r_n = C_{\lambda'}(F) - r_n \ . \tag{61}$$

Let $\lambda \in \Lambda_n$. Thanks to (58) and (59), there exists $\lambda' \subset \lambda$ such that

$$C_{\lambda'}(F) = e_\lambda(F) \geq \frac{1}{(2^{n+1}((n+1)\log 2)^2)^{\frac{1}{q}}} = 2r_n \ .$$

Thus, by (61), for such λ' we have

$$|C_{\lambda'}(f)| \geq C_{\lambda'}(F) - r_n = e_\lambda(F) - r_n \geq \frac{1}{2} e_\lambda(F).$$

Hence

$$e_\lambda(f) \geq |C_{\lambda'}(f)| \geq \frac{1}{2}e_\lambda(F).$$

□

Thanks to (60)

$$\forall\ f \in A_t^q \qquad \tilde{d}_f^+ \geq \tilde{d}_F^+ \ . \tag{62}$$

This result together with Proposition 4 and (62) yield (29). □

Let us now deduce Corollary 1.

- If $0 < t < q$ then for all $f \in A_t^q$, the increasing part of $\tilde{d}_f$ corresponds to $\alpha \in [0, 1/q]$, on which it has increasing visibility, and

$$\forall\ \alpha \in (0, 1/q] \quad \tilde{d}_f(\alpha) = \alpha q > \alpha t > d_f^\nu(\alpha) \ .$$

- If $0 < q \leq t$ then for all $f \in A_{q,t}$, the increasing part of $\tilde{d}_f$ corresponds to $\alpha \in [0, 1/t]$, on which it has increasing visibility, and

$$\forall\ \alpha \in [0, 1/t] \quad \tilde{d}_f(\alpha) = \alpha t = d_f^\nu(\alpha) \ .$$

The generic set $A_{q,t}$ (resp. A_q^t) is the same as the one (resp. is a correctly reduced generic set of the one because r_n was $1/(2(n(\log n)^2)^{1/q}))$ of Jaffard and Meyer [27], in which they computed the Hölder spectrum for functions when moreover $0 < q \leq 1$. (Note that, for $q > 1$, it was proved (see [27]) that quasi-all functions in $B_t^{m/t,q}(\mathbb{R}^m)$ are not locally bounded, so Hölder spectra are meaningless). This leads to Theorem 3 which confirms the effectiveness of the wavelet leader profile method compared to the sunny wavelet profile method.

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