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Special functions associated with complex reflection groups

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Abstract In this paper, we first review the theory of Dunkl operators for complex reflection groups and then the theory of hyper-Bessel functions, which are a particular case of Meijer's G-function and satisfy a higher order differential equation. Then we show that there exists a close relation between both theories. In fact, the components of the eigenfunctions of a Dunkl operator for a complex reflection group in the rank one case can be expressed in terms of hyper-Bessel functions.

Keywords Special functions · Meijer's G-function

Mathematics Subject Classification (2000) 33E30 · 33C52

1 Introduction

Dunkl theory generalizes the theory of special functions of one or several variables, which leads to generalizations of classical Fourier analysis and builds up the framework for a theory of special functions and integral transforms. This theory started 20 years ago with Dunkl, Heckman, and Cherednik. They showed that Weyl group invariant special functions associated with root systems can be obtained by symmetrization of certain special functions in several variables which are not Weyl group invariant, but which are in a sense more simple. These non-invariant special functions are joint eigenfunctions of Dunkl type operators. Roughly speaking, these are commuting differential or difference operators with reflection terms, associated to some

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finite reflection group, which can be a Weyl group or Coxeter group, or a complex reflection group [5].

The theory of special functions associated with real reflections and their related harmonic analysis has been intensively developed, but the one related to complex reflection group needs more study. In this work, we investigate in the rank one case some particular cases of complex reflection Dunkl operators and their relationship to the class of hyper-Bessel functions, Meijer's G-function, and H-function. Moreover, we show that the symmetrization of the eigenfunctions of complex reflection Dunkl operators are solution of higher order differential equations.

The specialization of this theory to the case of rank one has its own interest because everything can be done there in a much more explicit way, and new results for special functions in one variable can be obtained. In the rank one case, the complex reflection groups are cyclic groups of the form

$$G = \langle 1, \varepsilon, \dots, \varepsilon^{m-1} \rangle, \quad \varepsilon = e^{\frac{2i\pi}{m}}.$$

Dunkl operator $T(k)$ related to G is given by

$$T(k)f(z) := \frac{df(z)}{dz} + \sum_{i=1}^{m-1} \frac{k_i}{z} \sum_{j=0}^{m-1} \varepsilon^{-ij} f(\varepsilon^j z), \tag{1.1}$$

which is a particular case of the more general differential-complex reflections operator

$$T_A f(z) := \frac{df(z)}{dz} + \sum_{i=1}^{m-1} \frac{A_i'(z)}{A_i(z)} \sum_{j=0}^{m-1} \varepsilon^{-ij} f(\varepsilon^j z), \tag{1.2}$$

where A_i, $1 \le i \le m-1$ are real functions satisfying

$$A_i(\varepsilon z) = A_i(z). \tag{1.3}$$

We will prove in Sect. 4 that the mth power $T^m(k)$ of the operator (1.1) acts on G-invariant functions as the hyper-Bessel operator

$$B_m := \prod_{j=1}^{m-1} \left(\frac{d}{dz} + \frac{m\nu_j + m - j}{z} \right) \frac{d}{dz}. \tag{1.4}$$

The remaining sections of this paper are organized as follows. In Sect. 2, we first recall notations and some results for Dunkl operators and we establish a new representation for the complex Dunkl operator by using circular matrices. In Sect. 3, we discuss various types of generalizations of Bessel functions which can be found in the literature. In Sects. 4 and 5, we compute the eigenfunctions of the complex Dunkl operator related to the groups $G(m,1,1)$ and $\mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_N\mathbb{Z}$ in terms of hyper-Bessel functions.

2 Dunkl operators for complex reflection groups

In $\mathbb{C}^N$, we consider the standard hermitian form

$$\langle z, w\rangle := \sum_{k=1}^{N} \overline{z_k} w_k.$$

Let $U(N)$ be the group of unitary transformations of $\mathbb{C}^N$. The basic ingredient in the Dunkl theory are finite complex reflections acting on $\mathbb{C}^N$. An element $s \in U(N)$ is called a complex reflection if s has finite order and $H_s := \mathrm{Ker}(s - Id)$ is a hyperplane in $\mathbb{C}^N$. Let s be a complex reflection then there exists a nonzero vector $w \in \mathbb{C}^N$ and mth primitive root of unity ε such that

$$s(z) := s_{w,\varepsilon}(z) = z - (1-\varepsilon)\frac{\langle z, w\rangle}{|w|^2} w, \tag{2.1}$$

so that the matrix of s_w is given by

$$(s)_{i,j} = \delta_{ij} - (1-\varepsilon)\frac{w_i w_j}{|w|^2},$$

where $|w| = \sqrt{\langle w, w\rangle}$.

If t is a unitary transform for $\mathbb{C}^N$, we have

$$t s_{v,\varepsilon} t^{-1} = s_{t(v),\varepsilon}.$$

A finite complex reflection group is a finite subgroup of $U(N)$ generated by complex reflections. Let $m, p \in N$ be such that $p|m$. The subgroup $G(m, p, N)$ of $U(N)$ consists of permutation matrices whose nonzero entries are mth roots of unity and the product of the nonzero entries is an (m/p)th root of unity. This subgroup is a complex reflection group (see [6]). Let $G \subset U(N)$ be a finite complex reflection group acting in its reflection representation $\mathbb{C}^N$. Denote by $\mathcal{A}$ the set of reflection hyperplanes of reflection of G and write G_H for the (pointwise) stabilizer of $H \in \mathcal{A}$ in G.

Each G_H is a cyclic subgroup of G of order $m_H \geq 2$. For $H \in \mathcal{A}$, fix $v_H \in \mathbb{C}^N$ with $H = \langle v_H\rangle^{\perp}$ and write s_H for the complex reflection s_{v_H,ε_H} where $\varepsilon_H = \exp(2i\pi/m_H)$. The characters of G_H form a cyclic group, generated by the restriction χ_H of the determinant to G_H. We will thus label the character group of G_H by

$$\widehat{G}_H = \{\chi_H^{-j};\ j = 0, \dots, m_H - 1\}.$$

Consider the natural action of $U(N)$ on a function $f : \mathbb{C}^N \to \mathbb{C}$ which is given by

$$g.f(z) = f(g^{-1}z), \quad g \in U(N).$$

We define

$$p_{H,i} := \frac{1}{m_H}\sum_{j=0}^{m_H-1} \chi_H^{-i}(s_H^j) s_H^{-j}, \quad i = 0, \dots, m_H - 1. \tag{2.2}$$

These obey

$$id = \sum_{i=0}^{m_H-1} p_{H,i}, \quad p_{H,i} p_{H,j} = \delta_{ij} p_{H,i}. \tag{2.3}$$

Then, the elements $p_{H,i}$ are idempotents which are generalizations of the primitive idempotents $(1-s)/2$ and $(1+s)/2$ for a real reflection s.

For every $C \in \mathcal{A}/G$, we choose a vector $k_C = {}^T(k_{C,1}, \dots, k_{C,m_C-1}, 0) \in \mathbb{C}^{m_C}$. Let $H \in \mathcal{A}$, we put

$$a_H := a_H(k) = \sum_{i=1}^{m_H-1} k_{H,i} p_{H,i}. \tag{2.4}$$

For every $H \in \mathcal{A}$, we denote by Ω_H the $n_H \times n_H$ matrix which is given by (see [2])

$$\Omega_H := \frac{1}{m_H^{1/2}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \chi_H^{-1}(s_H) & \chi_H^{-1}(s_H^2) & \cdots & \chi_H^{-1}(s_H^{m_H-1}) \\ 1 & \chi_H^{-2}(s_H) & \chi_H^{-2}(s_H^2) & \cdots & \chi_H^{-2}(s_H^{m_H-1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \chi_H^{-m_H+1}(s_H) & \chi_H^{-m_H+1}(s_H^2) & \cdots & \chi_H^{-m_H+1}(s_H^{m_H-1}) \end{pmatrix}. \tag{2.5}$$

For every $0 \le i, j \le n_H - 1$, we have

$$\chi_H^{-j}(s_H^i) = \chi_H^{-i}(s_H^j) = \overline{\chi_H^i(s_H^j)}.$$

Hence, Ω_H is a symmetric matrix.

Let $f : \mathbb{C}^N \to \mathbb{C}$ and $H \in \mathcal{A}$, we denote by $\Lambda_H(f)(z)$ the vector-valued function from $\mathbb{C}^N$ into $\mathbb{C}^{m_H}$, defined by

$$\Lambda_H(f)(z) := \begin{pmatrix} f(z) \\ f(s_H z) \\ \vdots \\ f(s_H^{m_H-1} z) \end{pmatrix}. \tag{2.6}$$

Let $w \in \mathbb{C}^N$. The Dunkl operator is a differential-complex reflection operator associated to G defined by [6]

$$\begin{aligned} T_w(f)(z) &:= \partial_w f(z) + \sum_{H\in\mathcal{A}} \frac{\langle w, v_H\rangle_{m_H}}{\langle z, v_H\rangle_{m_H}} a_H(f)(z), \\ &= \partial_w f(z) + \sum_{H\in\mathcal{A}} \frac{1}{m_H} \frac{\langle w, v_H\rangle_{m_H}}{\langle z, v_H\rangle_{m_H}} \sum_{i=1}^{m_H-1} \sum_{j=0}^{m_H-1} k_i \chi_H^{-j}(s_H^i) f(s_H^i z), \end{aligned} \tag{2.7}$$

where ∂_w denotes the directional derivative corresponding to $w \in \mathbb{C}^N$.

Proposition 2.1

$$T_w f(z) = \partial_w f(z) + \sum_{H\in\mathcal{A}} \frac{\langle w, v_H\rangle_{m_H} \langle \Omega_H \Lambda_H f, k_H\rangle_{m_H}}{\langle z, v_H\rangle_{m_H}}.$$

Proof A simple calculation shows that

$$a_H(f)(z) = \big\langle \Omega_H \Lambda_H(f)(z), k_H\big\rangle_{m_H}.$$

Hence,

$$T_w f(z) = \partial_w f(z) + \sum_{H\in\mathcal{A}} \frac{\langle w, v_H\rangle_{m_H} \langle \Omega_H \Lambda_H f, k_H\rangle_{m_H}}{\langle z, v_H\rangle}. \qquad \square$$

We denote by $\mathcal{P} := \mathbb{C}[z]$ the $\mathbb{C}$-algebra of polynomial functions of N variables $z = (z_1, \dots, z_N)$. It has the natural grading

$$\mathcal{P} = \bigoplus_{n\in\mathbb{N}} \mathcal{P}_n,$$

where $\mathcal{P}_n$ is the subspace of homogeneous polynomials of (total) degree n.

Lemma 2.2

1. *If $f \in \mathcal{E}_N(\mathbb{C})$ then $T_w f \in \mathcal{E}_N(\mathbb{C})$.*
2. *The Dunkl operator T_w is a homogeneous differential-difference operator of degree -1 on $\mathcal{P}$, that is, $T_w p \in \mathcal{P}_{n-1}$ for $p \in \mathcal{P}_n$.*

Proof This follows immediately from the fact that for $i = 1, \dots, m_H - 1$,

$$\begin{aligned} p_{H,i}(f)(z) &= \frac{1}{m_H} \sum_{j=0}^{m_H-1} \chi^{-j}\big(s_H^j\big) f\big(s^j z\big) \\ &= \langle z, v_H\rangle \Bigg(- \sum_{j=0}^{m_H-1} \frac{(1-\varepsilon_H^j)\chi^{-j}(s_H^i)}{m_H |v_H|^2} \\ &\quad \times \int_0^1 \partial_{v_H} f\bigg(z - t\big(1-\varepsilon_H^j\big) \frac{\langle z, v_H\rangle}{|v_H|^2} v_H \bigg)\, dt \Bigg). \end{aligned} \qquad \square$$

The following proposition follows by an easy calculation.

Proposition 2.3

1. *$g \circ T_w \circ g^{-1} = T_{gw}$ for all $g \in G$.*
2. *If $f \in \mathcal{E}_N(\mathbb{C})$ is G-invariant then $T_w f = \partial_w$.*
3. *If $f, g \in \mathcal{E}_N(\mathbb{C})$, and least one of them is G-invariant, then*

$$T_w(fg) = T_w(f)g + f T_w(g).$$

Theorem 2.4 [6] *Let G a finite complex reflection group. Then for all* $z, w \in \mathbb{C}^N$

$$T_w T_z = T_z T_w.$$

Example 2.1 (Coxeter groups) Let G be a finite Coxeter group and R be a fixed root system associated to G. Under the standard embedding $\mathbb{R}^N \subset \mathbb{C}^N$, we assume that R is a root system in $\mathbb{C}^N$, so that we can define the positive root system $R_+ \subset \mathbb{R}^N \subset \mathbb{C}^N$. Moreover, for any real reflection $s \in O(N)$, we can regard s as complex reflection. In particular, the groups $G(1,1,N)$, $G(2,1,N)$, and $G(2,2,N)$ are the Coxeter groups of types A_{N-1}, B_N, and D_N, respectively. In this case, each G_H is generated by a real reflection s_H of order $m_H = 2$, the corresponding idempotents are given by

$$e_{H,0} = (1 + s_H)/2 \quad \text{and} \quad e_{H,1} = (1 - s_H)/2,$$

and the related Dunkl operator is defined by [5]

$$T_w f(z) = \partial_w f(z) + \sum_{\alpha \in R_+} k(\alpha) \langle \alpha, w \rangle \frac{f(z) - f(s_\alpha z)}{\langle z, \alpha \rangle}.$$

Example 2.2 (The group $G(m,1,N)$) Let $\varepsilon := \varepsilon_m = e^{\frac{2i\pi}{m}}$. The group $G(m,1,N)$, consists of the $N \times N$ permutation matrices with the nonzero entries being powers of ε. The group is generated by the transpositions $(i, i+1)$, $i = 1, \ldots, N-1$, and by the complex reflection s which is defined by

$$s_i z = (z_1, \ldots, \varepsilon z_i, \ldots, z_N).$$

The symmetric group S_N is obviously a subgroup of $G(m,1,N)$. In this notation, the Dunkl operator is given by [6]

$$T_i = \frac{\partial}{\partial x_i} + k_0 \sum_{j \neq i} \sum_{r=0}^{m-1} \frac{1 - s_i^{-r}(i,j) s_i^r}{x_i - \varepsilon^r x_j} + \sum_{j=1}^{m-1} k_j \sum_{r=0}^{m-1} \frac{\varepsilon^{-rj} s_i^r}{x_i}.$$

The classical real reflection groups occur as the special cases $A_{N-1} = S_N = G(1,1,N)$, $BC_N = G(2,1,N)$, and $D_N = G(2,2,N)$.

2.1 Decomposition of functions with respect to the cyclic group

Let G be a finite complex reflection group, denote by $\mathcal{A}$ the set of reflection hyperplanes of reflections in G, and write G_H for the stabilizer of $H \in \mathcal{A}$ in G.

Definition 2.1 Let $H \in \mathcal{A}$ and $0 \leq j \leq m_H - 1$. A function $f : \mathbb{C}^N \to \mathbb{C}$ is called of type j with respect to H if

$$f(s_H z) = \chi_H^j(s_H) f(z)$$

holds for every $z \in \mathbb{C}^N$.

Lemma 2.5 *Let $H \in \mathcal{A}$ and $f : \mathbb{C}^N \to \mathbb{C}$. Then, f can be decomposed uniquely in the form*

$$f = \sum_{j=0}^{m_H - 1} f_{H,j}, \tag{2.8}$$

where the component function $f_{H,j}$ is of type j and given by

$$f_{H,i}(z) = p_{H,i}(f)(z) = \left\langle \Omega_H \Lambda_H(f)(z), e_i \right\rangle_{m_H}, \tag{2.9}$$

where $\{e_i\}$ is the standard canonical basis of $\mathbb{C}^{m_H}$ and $< \cdot, \cdot >_{m_H}$ is the canonical hermitian product in $\mathbb{C}^{m_H}$.

Let $\mathcal{E}_N(\mathbb{C})$ be the $\mathbb{C}$-algebra of entire functions in $\mathbb{C}^N$, and we denote by $\mathcal{E}_{H,j}(\mathbb{C})$ the subspace of $\mathcal{E}_N(\mathbb{C})$ of functions of type j with respect to the hyperplane H. Of course, we have

$$\mathcal{E}_N(\mathbb{C}) = \bigoplus_{j=0}^{m_j - 1} \mathcal{E}_{H,j}(\mathbb{C}).$$

Example 2.3 1. Let $\kappa := \kappa_m = e^{\frac{i\pi}{m}}$ and let $G = \langle 1, \varepsilon, \ldots, \varepsilon^{m-1} \rangle$ be a cyclic group where $\varepsilon = e^{\frac{2i\pi}{m}}$. The hyper-trigonometric functions are a components of the exponential function $e^{\kappa z}$ with respect to G. Also these functions can be considered as a generalization of the elementary trigonometric functions $\cos(z)$ and $\sin(z)$ (see [7, 12]).

The m-cosine is given by

$$\begin{aligned} \cos_m(z) = \sin_{m,m}(z) :&= \frac{1}{m} \sum_{j=0}^{m-1} e^{i\kappa\varepsilon^j z} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{nm}}{(nm)!}. \end{aligned} \tag{2.10}$$

The m-sine functions of type l with $1 \leq l \leq m-1$ are given by

$$\begin{aligned} \sin_{m,l}(z) &= \frac{1}{m\kappa^l} \sum_{j=0}^{m-1} \varepsilon^{lj} e^{i\kappa\varepsilon^j z} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{nm+l}}{(nm+l)!}. \end{aligned}$$

The function $y(z) = \cos_m(\lambda z)$ is the unique C^∞-solution of the system

$$\begin{cases} y^{(m)}(z) = -\lambda^m y(z), \\ y(0) = 1, \qquad y^{(1)}(0) = \cdots = y^{(m-1)}(0) = 0. \end{cases}$$

Furthermore, we have

$$\frac{d^k}{dz^k}\sin_{m,l}(z)=\begin{cases}\sin_{m,l-k}(z) & \text{for } 1\le k\le l-1,\\ \cos_m(z) & \text{for } k=l,\\ -\sin_{m,m+l-k}(z) & \text{for } l\le k.\end{cases} \tag{2.11}$$

2. The components of type j $(0\le j\le n-1)$ of the generalized hypergeometric series ${}_pF_q$ with respect to G is given by (see [1])

$$\frac{(a_1)_j\cdots(a_p)_j}{(b_1)_j\cdots(b_q)_j}\frac{z^j}{j!}\,{}_pF_q\left(\begin{matrix}\Delta(n,a_1+j),\ldots,\Delta(n,a_p+j)\\ \Delta^*(n,j+1),\Delta(n,b_1+j),\ldots,\Delta(n,b_p+j)\end{matrix}\middle|\frac{z^n}{n^{(1-p+q)n}}\right),$$

where $\Delta(n,a)$ is the following set

$$\frac{a}{n},\frac{a+1}{n},\ldots,\frac{a+n-1}{n},$$

and $\Delta^*(n,j+1)$ represents the fact that the denominator $\frac{n}{n}$ is always omitted.

3 Generalizations of Bessel functions

3.1 The Bessel functions

The normalized Bessel function $j_\alpha(z)$ is defined by

$$j_\alpha(z):=\sum_{j=0}^{\infty}\frac{(-\frac{1}{4}z^2)^j}{(\alpha+1)_j\,j!};\quad z\in\mathbb{C}. \tag{3.1}$$

Here we use the notation of the shifted factorial:

$$(a)_j:=a(a+1)\cdots(a+j-1)\quad(j=1,2,\ldots);\quad (a)_0:=1.$$

The function $j_\alpha(x)$ is related to the Bessel functions J_α (see [13]) by

$$j_\alpha(z):=\frac{2^\alpha\Gamma(\alpha+1)}{z^\alpha}J_\alpha(z).$$

The function $y(z)=j_\alpha(\lambda z)$ is the unique C^∞-solution of the problem

$$B_2y(z)=-\lambda^2y(z),\quad y(0)=1,\quad y'(0)=0,$$

where

$$B_2:=\frac{d^2}{dz^2}+\frac{2\alpha+1}{z}\frac{d}{dz}. \tag{3.2}$$

The cases $\alpha=-\frac{1}{2}$ and $\alpha=\frac{1}{2}$ yield the functions

$$j_{-1/2}(z)=\cos(z),\qquad j_{1/2}(z)=\frac{\sin(z)}{z}.$$

Then we get the elementary formulas

$$e^{i\lambda z} = j_{-1/2}(\lambda z) + i\lambda z j_{1/2}(\lambda z) \quad \text{and} \quad \frac{d}{dz} e^{i\lambda z} = i\lambda e^{i\lambda z}.$$

Dunkl [5] generalized the operator $\frac{d}{dx}$ to a mixture of a differential and a real reflection operator

$$T = T(\alpha) := \frac{d}{dz} + \frac{\alpha + 1/2}{z}(1 - s), \tag{3.3}$$

where

$$(sf)(z) := f(-z).$$

The generalized exponential function is defined by

$$\mathcal{E}_\alpha(i\lambda z) := j_\alpha(\lambda z) + i\lambda z j_{\alpha+1}(\lambda z).$$

Then it follows immediately from well-known differential recurrence formulas for Bessel functions that

$$T\mathcal{E}_\alpha(i\lambda z) = i\lambda \mathcal{E}_\alpha(i\lambda z).$$

Consider the Dunkl type operator $T_{A,\tau}$,

$$T_{A,\tau} =: \frac{d}{dz} + \frac{1}{2}\left(\frac{A'}{A} - 2\tau\right)(1 - s) - \tau, \tag{3.4}$$

where A is a real function and τ a real number.

The simplest examples of the operator $T_{A,\tau}$ are provided by

1. The Dunkl operator for which $(A(z) = |z|^{2\alpha+1}, \tau = 0)$;
2. The Dunkl–Heckman operator $(A(z) = \sinh^{k_1+k_2/2}(|z|) \cosh^{k_2/2}(|z|), \tau = 0)$

$$T^{(k_1,k_2)} := \frac{d}{dz} + \left(k_1 \frac{1 + e^{-2z}}{1 - e^{-2z}} + 2k_2 \frac{1 + e^{-2z}}{1 - e^{-2z}}\right)(1 - s). \tag{3.5}$$

3. The Cherednik operator $(A(z) = \sinh^{2(k_1+k_2)}(|z|) \cosh^{2k_2}(|z|), \tau = k_1 + 2k_2)$ where

$$Y^{(k_1,k_2)} := \frac{d}{dz} + \left(\frac{2k_1}{1 - e^{-2z}} + \frac{4k_2}{1 - e^{-2z}}\right)(1 - s) - (k_1 + 2k_2). \tag{3.6}$$

A simple computation shows that the square of $\Delta_{A,\tau}$ is given by

$$T_{A,\tau}^2 = \frac{d^2}{dz^2} + \frac{A'}{A}\frac{d}{dz} + \left(\frac{1}{2}\left(\frac{A'}{A}\right)' + \tau\frac{A'}{A} + 2\tau^2\right)(1 - s) + \tau^2. \tag{3.7}$$

Thus, on even functions the square of the operator $T_{A,\tau}$ acts as the following second order differential operator

$$L_{A,\tau} := \frac{d^2}{dz^2} + \frac{A'}{A}\frac{d}{dz} + \tau. \tag{3.8}$$

3.2 The hyper-Bessel functions

The hyper-Bessel differential operator, or a Bessel type differential operator, is a singular linear differential operator of arbitrary order $m \geq 2$ of the form (see [12])

$$B_m := \frac{d^m}{dz^m} + \frac{a_1}{z}\frac{d^{m-1}}{dz^{m-1}} + \cdots + \frac{a_{m-1}}{z^{m-1}}\frac{d}{dz}, \tag{3.9}$$

with arbitrary real numbers $a_1, \ldots, a_{m-1}$.

The operator B_m can be written in the form

$$B_m = z^{-m+1}\prod_{k=1}^{m-1}\left(z\frac{d}{dz} + m\nu_k + 1\right)\frac{d}{dz}, \tag{3.10}$$

where the coefficients a_{m-k} are given by

$$a_{m-k} = \sum_{j=1}^{k}\frac{(-1)^j}{j!(k-j)!}\prod_{s=1}^{m-1}(m\nu_s + k - j), \quad k = 0, 1, \ldots, m-1. \tag{3.11}$$

By the following formula

$$\left(z\frac{d}{dz} + \alpha\right)z^{\beta} = z^{\beta}\left(z\frac{d}{dz} + \alpha + \beta\right),$$

the operator B_m takes the form:

$$B_m = \prod_{j=1}^{m-1}\left(\frac{d}{dz} + \frac{m\nu_j + m - j}{z}\right)\frac{d}{dz}. \tag{3.12}$$

The simplest higher order hyper-Bessel operator is the operator of m-fold differentiation

$$\frac{d^m}{dz^m} = z^{-m}\left(z\frac{d}{dz}\right)\left(z\frac{d}{dz} - 1\right)\cdots\left(z\frac{d}{dz} - m + 1\right).$$

For $m = 2$ and $a_1 = 2\alpha + 1$, the hyper-Bessel operator generalizes the well known second order differential operator of Bessel B_2 defined in (3.2).

In 1953, Delerue [3] introduced for the first time the hyper-Bessel functions $J_\nu(x)$ with a vector index $\nu = (\nu_1, \ldots, \nu_{m-1}) \in \mathbb{R}^{m-1}$ satisfying for $j = 1, \ldots, m-1$, $\nu_j > -1$, that is,

$$J_\nu(z) := \frac{(\frac{z}{m})^{|\nu|}}{\Gamma(\nu + \mathbf{1})}\,{}_0F_{m-1}\left(\begin{matrix}\emptyset\\ \nu + \mathbf{1}\end{matrix}\middle| -\left(\frac{z}{m}\right)^m\right), \tag{3.13}$$

where

$$|\nu| = \nu_1 + \cdots + \nu_{m-1},$$

$$\nu + \mathbf{n} = (\nu_1 + n, \ldots, \nu_{m-1} + n) \quad (n \in \mathbb{N}),$$

$$\Gamma(\nu) = \Gamma(\nu_1) \times \cdots \times \Gamma(\nu_{m-1}).$$

The normalized hyper-Bessel function of index ν is defined by [11]

$$\mathcal{J}_\nu(z) := \left(\frac{z}{m}\right)^{-|\nu|} \Gamma(\nu+\mathbf{1}) J_\nu(z)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\nu+\mathbf{1})}{n! \Gamma(\nu+\mathbf{n}+\mathbf{1})} \left(\frac{z}{m}\right)^{nm}. \tag{3.14}$$

The function $\mathcal{J}_\nu(\lambda z)$ is the unique C^∞-solution of the following problem [11]

$$\begin{cases} B_m(f)(z) = -\lambda^m f(z), \\ f(0) = 1, \qquad f^{(1)}(0) = \cdots = f^{(m-1)}(0) = 0. \end{cases} \tag{3.15}$$

From Corollary 2 in [11] and (3.14), we obtain the following differential recurrence relations for the normalized hyper-Bessel functions $\mathcal{J}_\nu(x)$

$$\frac{d}{dz}\mathcal{J}_\nu(z) = -\frac{(\frac{z}{m})^{m-1}}{(\nu_1+1)\cdots(\nu_{m-1}+1)} \mathcal{J}_{\nu+\mathbf{1}}(z), \tag{3.16}$$

$$\left(\frac{d}{dz} + \frac{m\nu_k}{z}\right)\mathcal{J}_\nu(z) = \frac{m\nu_k}{z}\mathcal{J}_{\nu-e_k}(z), \tag{3.17}$$

where e_j $(1 \le j \le m-1)$ is the standard basis of $\mathbb{R}^{m-1}$.

The normalized hyper-Bessel function has the Poisson integral representation [4]

$$\mathcal{J}_\nu(z) = \frac{m^{3/2}\Gamma(\nu+\mathbf{1})}{(2\pi)^{(m-1)/2}} \int_0^1 G^{m-1,0}_{m-1,m-1}\left(\begin{matrix} \nu_1, \nu_2, \ldots, \nu_{m-1} \\ -\frac{1}{m}, \ldots, -\frac{m-1}{m} \end{matrix} \middle| t^m \right) t^{m-1} \cos_m(tz)\, dt, \tag{3.18}$$

where the function $\cos_m$ is defined in (2.10), the Meijer's G-function $G^{m,n}_{p,q}(z)$ is given by means of the contour integral in the complex plane

$$G^{m,n}_{p,q}(z) := G^{m,n}_{p,q}\left(\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix} \middle| z \right) = \frac{1}{2i\pi} \int_{\mathcal{C}} \mathfrak{G}^{m,n}_{p,q}(s) z^s\, ds,$$

and

$$\mathfrak{G}^{m,n}_{p,q}(s) := \frac{\Pi_{k=1}^m \Gamma(b_k - s) \Pi_{k=1}^n \Gamma(1 - a_k + s)}{\Pi_{k=m+1}^q \Gamma(1 - b_k + s) \Pi_{k=n+1}^p \Gamma(a_k - s)}.$$

Here $\mathcal{C}$ is a suitable contour in $\mathbb{C}$; m, n, p, q are integers such that $0 \le m \le q$, $0 \le n \le q$; the parameters a_k and b_k are complex numbers for which

$$b_k + l \neq a_j - l' - 1; \quad j = 1, \ldots, p;\ k = 1, \ldots, q;\ l, l' = 0, 1, 2, \ldots.$$

The Meijer's G-function plays an important role in the theory of special functions because almost all the special functions, as well as the elementary functions, can be represented as G-functions. In particular, the generalized hypergeometric series ${}_pF_q$ is related to Meijer's G-function by

$${}_pF_q\left(\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix} \middle| z \right)$$

$$= \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{k=1}^{p}\Gamma(a_k)} G_{p,p+q}^{1,p}\left(\begin{matrix}1-a_1, 1-a_2, \dots, 1-a_p\\ 0, 1-b_1, 1-b_2, \dots, 1-b_q\end{matrix}\middle| -z\right).$$

Furthermore, the hyper-Bessel can be represented as a product of m-integrals

$$\mathcal{J}_\nu(z) = \frac{(\frac{m}{2})^{|\nu|+m-1/2}\Gamma(\nu+\mathbf{1})}{(2\pi)^{(m-1)/2}\Gamma(\nu+\mathbf{1}/2)} \times \int_0^1 \cdots \int_0^1 \prod_{k=1}^{m-1}\left(1-t_k^m\right)^{\nu_k-k/m} t_k^{k-1}\cos_m(zt_1\cdots t_{m-1})\,dt_1\cdots dt_{m-1}, \tag{3.19}$$

3.3 G-Bessel function

Let $n = 1, 2, \dots$, and $m = 0, 1, \dots$. The G-Bessel function is defined by [10]

$$G(z) := G_{0,n}^{n,0}\left(\begin{matrix}\emptyset\\ b_1+m, b_2+m, \dots, b_q+m\end{matrix}\middle| z\right). \tag{3.20}$$

The function $y(z) = G_{p,q}^{m,n}(z)$ satisfies the linear ordinary differential equation

$$\left[(-1)^{p-m-n} z \prod_{j=1}^{p}\left(z\frac{d}{dz} - a_j + 1\right) - \prod_{k=1}^{q}\left(z\frac{d}{dz} - b_k\right)\right] y(z) = 0.$$

In particular, for $n, m \geq 1$, the function $G(z)$ defined in (3.20) is an eigenfunction of the hyper-Bessel type operator

$$\Delta_{n,m} G(z) = (-1)^m G(z),$$

where

$$\Delta_{n,m} := z^{-m} \prod_{k=1}^{n}\prod_{j=1}^{m-1}\left(z\frac{d}{dz} - b_k - m + j\right).$$

4 Complex reflection Dunkl operator associated to $G(m, 1, 1)$

In the one-dimensional case ($N = 1$), we take $p = 1$. The corresponding reflection group $G(m, 1, 1)$ ($m \geq 2$) is a cyclic group $\mathbb{Z}/m\mathbb{Z}$ acting on $\mathbb{C}$ by multiplication by the mth roots of unity $\varepsilon := \varepsilon_m = e^{\frac{2i\pi}{m}}$. In this case, we have only one reflection "hyperplane", with multiplicities $k = (k_1, \dots, k_{m-1}, 0)$ where $k_j = m\nu_j + m - j$ and the corresponding Dunkl operator $T(k)$ is given by

$$T(k)f(z) = \frac{df(z)}{dz} + \sum_{i=1}^{m-1}\frac{k_i}{z}\sum_{j=0}^{m-1}\varepsilon^{-ij} f\left(\varepsilon^j z\right). \tag{4.1}$$

These operators and their connection to hyper-Bessel functions are also investigated in [8].

Theorem 4.1 *Under the condition*

$$k_j = m\nu_j + m - j \geq 0, \quad j = 1, \dots, m-1, \tag{4.2}$$

the following system

$$\begin{cases} T(k)f(z) = \kappa\lambda f(z), \\ f(0) = 1. \end{cases} \tag{4.3}$$

has a unique C^∞*-solution which is given by*

$$\begin{aligned} \mathcal{D}_\nu(\lambda, z) &= \sum_{j=1}^{m} (\kappa\lambda)^{-j} A_j \mathcal{J}_\nu(\lambda z) \\ &= \mathcal{J}_\nu(\lambda z) + \sum_{j=1}^{m-1} \frac{(\kappa\lambda)^j}{m^j(\nu_1+1)\cdots(\nu_{m-j}+1)} \mathcal{J}_{(\nu_1+1,\dots,\nu_j+1,\nu_{j+1},\dots,\nu_{m-1})}(\lambda z), \end{aligned} \tag{4.4}$$

where

$$A_m = 1, \qquad A_{m-1} = \frac{d}{dz}, \qquad A_j = \prod_{k=j+1}^{m-1} \left(\frac{d}{dz} + \frac{m\nu_k + m - k}{z} \right) \frac{d}{dz}, \quad 2 \leq j \leq m. \tag{4.5}$$

Proof Let f be a solution of the system (4.3). We decompose f as

$$f = \sum_{j=1}^{m} f_j,$$

where the function f_j is of type j. Then the system (4.3) is equivalent to

$$\begin{cases} \sum_{j=1}^{m} f_j'(z) + \sum_{j=1}^{m-2} \frac{m\nu_j+m-j}{z} f_j(z) = \kappa\lambda \sum_{j=1}^{m} f_j(z), \\ f_m(0) = 1, \qquad f_1(0) = \cdots = f_{m-1}(0) = 0. \end{cases} \tag{4.6}$$

For $j = 1, \dots, m$, the functions f_j' and $z^{-1} f_j$ are of type $j-1$, and the functions f_1', $z^{-1} f_1$ are of type m.

Hence, we can write the system (4.6) in the following equivalent form

$$\begin{cases} f_1' + \frac{m\nu_1+m-1}{z} f_1 = \kappa\lambda f_m, \\ f_2' + \frac{m\nu_2+m-2}{z} f_2 = \kappa\lambda f_1, \\ \cdots \\ f_{m-1}' + \frac{m\nu_{m-1}+1}{z} f_{m-1} = \kappa\lambda f_{m-2}, \\ f_m' = \kappa\lambda f_{m-1}, \\ f_m(0) = 1, \qquad f_1(0) = \cdots = f_{m-1}(0) = 0. \end{cases}$$

Therefore, the function f_m satisfies

$$\begin{cases} \prod_{j=1}^{m-1}(\frac{d}{dz}+\frac{m\nu_j+m-j}{z})\frac{d}{dz}f_m=-\lambda^m f_m, \\ f_m(0)=1, \qquad f_m^{(1)}(0)=\cdots=f_m^{(m-1)}(0)=0. \end{cases} \tag{4.7}$$

Thus,

$$f_m=\mathcal{J}_\nu(\lambda z),$$
$$f_j(z)=(\kappa\lambda)^{j-m}A_j\mathcal{J}_\nu(\lambda z) \quad (1\le j\le m-1),$$

where

$$\begin{aligned} A_m&=1, \qquad A_{m-1}=\frac{d}{dz}, \\ A_j&=\prod_{k=j+1}^{m-1}\left(\frac{d}{dz}+\frac{m\nu_k+m-k}{z}\right)\frac{d}{dz}, \quad 1\le j\le m-2. \end{aligned} \tag{4.8}$$

More precisely, one has

$$f(z)=\mathcal{J}_\nu(\lambda z)+\sum_{j=1}^{m-1}\frac{(\kappa\lambda)^j}{m^j(\nu_1+1)\cdots(\nu_{m-j}+1)}\mathcal{J}_{(\nu_1+1,\dots,\nu_j+1,\nu_{j+1},\dots,\nu_{m-1})}(\lambda z).$$
□

Proposition 4.2

1. *If f is of type j, with $1\le j\le m$, then $T(k)$ is of type $j-1$ and*

$$T(k)f(z)=\frac{df(z)}{dz}+\frac{k_j}{z}f(z).$$

2. *If f is of type m, then*

$$T^m(k)f=B_m f,$$

where

$$B_m=\prod_{j=1}^{m-1}\left(\frac{d}{dz}+\frac{k_j}{z}\right)\frac{d}{dz}.$$

4.1 Integral representation

In [4], Dimovski and Kiryakova studied the generalized Riemann–Liouville transform $\mathcal{R}_{\nu,m}$, which is defined by

$$\begin{aligned} \mathcal{R}_{\nu,m}f(z) &:= \frac{m^{1/2}\Gamma(\nu+\mathbf{1})}{(2\pi)^{(m-1)/2}}\int_0^1 G_{m-1,m-1}^{m-1,0}\left(\begin{matrix}\nu_1,\nu_2,\dots,\nu_{m-1}\\ -\frac{1}{m},\dots,-\frac{m-1}{m}\end{matrix}\middle| t^m\right)t^{m-1}f(tz)\,dt \\ &= \frac{(\frac{m}{2})^{|\nu|+m-1/2}\Gamma(\nu+\mathbf{1})}{(2\pi)^{(m-1)/2}\Gamma(\nu+\mathbf{1}/2)} \end{aligned}$$

$$\times \int_0^1 \cdots \int_0^1 \prod_{k=1}^{m-1} \left(1 - t_k^m\right)^{\nu_k - k/m} t_k^{k-1} f(zt_1 \cdots t_{m-1})\, dt_1 \cdots dt_{m-1}. \tag{4.9}$$

The operator $\mathcal{R}_\nu^m$ intertwines the hyper-Bessel operator B_m operator defined by (3.10) and the mth differential operator $\frac{d^m}{dz^m}$ (see [9])

$$B_m \circ \mathcal{R}_{\nu,m} = \mathcal{R}_{\nu,m} \circ \frac{d^m}{dz^m}. \tag{4.10}$$

For $m = 2$, the transform $\mathcal{R}_{\nu,m}$ is reduced to the so-called Riemann–Liouville transform

$$\mathcal{R}_{\nu,2} f(z) = \frac{2\Gamma(\nu+1)}{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})} \int_0^1 \left(1 - t^2\right)^{\nu - \frac{1}{2}} f(zt)\, dt. \tag{4.11}$$

From (3.18), we get

$$\mathcal{J}_\nu(\lambda z) = \mathcal{R}_{\nu,m}\left(\cos_m(\lambda \cdot)\right)(z) \tag{4.12}$$

Let consider the operator V_m defined by

$$V_m := \sum_{j=1}^{m} A_j \circ \mathcal{R}_{\nu,m} \circ I^{m-j} \circ p_j, \tag{4.13}$$

where the operator p_j is the projection operator defined in (2.2) and I is given by

$$I(f)(z) = \int_0^z f(t)\, dt.$$

The operator I is the right inverse of the derivative operator $\frac{d}{dz}$:

$$\frac{d}{dz} \circ I = 1.$$

Lemma 4.3 *For $n = 1, 2, \ldots,$ we have*

$$\left(I^n \circ \frac{d^n}{dz^n}\right)(f)(z) = f(z) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} z^k.$$

Theorem 4.4 *The hyper-Dunkl–Bessel function has the integral representation*

$$\mathcal{D}(\lambda, z) = V_m\left(e^{\kappa\lambda \cdot}\right)(z).$$

Proof The exponential function $e^{\kappa\lambda z}$ has the following decomposition with respect to the cyclic group of order m

$$e^{\kappa\lambda z} = \cos_m(\lambda z) + \sum_{j=1}^{m-1} \kappa^j \sin_{m,l}(\lambda z).$$

From the relation (2.11), we get

$$\sin_{m,j}(\lambda z) = \frac{1}{\lambda^{m-j}} \frac{d^{m-j}}{dz^{m-j}} \cos_m(\lambda z).$$

By Lemma 4.3, we can write

$$\begin{aligned} I^{m-j} \circ p_j\big(e^{\kappa\lambda z}\big) &= \frac{1}{(\kappa\lambda)^{m-j}} \left(I^{m-j} \circ \frac{d^{m-j}}{dz^{m-j}} \right) \cos_m(\lambda z) \\ &= \frac{1}{(\kappa\lambda)^{m-j}} \left(\cos_m(\lambda z) - \sum_{k=0}^{m-j-1} (-1)^k \frac{(\lambda z)^{km}}{(km)!} \right). \end{aligned}$$

Hence,

$$\begin{aligned} &\mathcal{R}_{\nu,m} \circ I^{m-j} \circ p_j\big(e^{\kappa\lambda\cdot}\big)(z) \\ &\quad = (\kappa\lambda)^{j-m} \left(\mathcal{J}_\nu(\lambda z) - \sum_{k=0}^{m-j-1} (-1)^k \frac{\Gamma(\nu+\mathbf{1})}{\Gamma(\nu+\mathbf{k}+\mathbf{1})k!} \left(\frac{\lambda z}{m}\right)^{km} \right). \end{aligned}$$

The order of differential operator A_j is equal to $m-j$ $(j=1,\dots,m)$, then

$$A_j \left(\sum_{k=0}^{m-j-1} (-1)^k \frac{\Gamma(\nu+\mathbf{1})}{\Gamma(\nu+\mathbf{k}+\mathbf{1})k!} \left(\frac{\lambda z}{m}\right)^{km} \right) = 0.$$

Thus,

$$V_m\big(e^{\kappa\lambda\cdot}\big)(z) = \sum_{j=1}^{m} (\kappa\lambda)^{j-m} A_j \mathcal{J}_\nu(\lambda z) = \mathcal{D}(\lambda, z).$$

□

5 Complex reflection Dunkl operator associated to $\mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_N\mathbb{Z}$

Let $e_1,\dots,e_N$ be the standard basis of $\mathbb{C}^N$. We denote by s_j $(1 \le j \le N)$ the refection with respect to the hyperplane perpendicular to e_j, that is to say, for every $z=(z_1,\dots,z_N) \in \mathbb{C}^N$,

$$\begin{aligned} s_j(z) &:= z - (1-\varepsilon_j) \frac{\langle z, e_j\rangle}{\|e_j\|} e_j \\ &= (z_1,\dots,z_{j-1},\varepsilon_j z_j, z_{j+1},\dots,z_N), \end{aligned} \tag{5.1}$$

where $\varepsilon_j = e^{i\frac{2\pi}{m_j}}$, and $m_j = 2,3,\dots$. Let G be the finite reflection group generated by $\{s_j : j=1,\dots,N\}$, so G is isomorphic to $\mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_N\mathbb{Z}$. The corresponding group has the relations

$$s_j s_k = s_k s_j \quad \text{and} \quad s_j^{m_j} = 1. \tag{5.2}$$

For $1 \le j \le N$ and $1 \le i \le m_j - 1$, let $\nu_{i,j}$ be real numbers satisfying $m_j \nu_{r,j} - m_j + r \ge 0$. Associated with these objects are the Dunkl operators T_j (for $j = 1, \dots, N$)

$$T_j f(z) = \frac{\partial_j f(z)}{\partial z_j} + \sum_{r=1}^{m_j-1} \frac{m_j \nu_{r,j} - m_j + r}{z_j} \sum_{i=0}^{m_j-1} \varepsilon_j^{-ri} f\big(s_j^i(z)\big).$$

Proposition 5.1 *For $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N$, the initial system problem*

$$T_j f(z) = \kappa_j \lambda_j f(z), \quad f(0) = 1, \quad j = 1, \dots, N,$$

has a unique C^∞-solution $\mathcal{D}_\nu^N(z, \lambda)$ called Dunkl kernel and given by

$$\mathcal{D}_\nu^N(\lambda, z) := \prod_{j=1}^{N} \mathcal{D}_{\nu_j}(\lambda_j, z_j),$$

where $\kappa_j = e^{i\frac{\pi}{m_j}}$, $\nu = (\nu_{i,j})$, $\nu_j = (\nu_{i,j})_{i=1}^{m_j}$, and $\mathcal{D}_{\nu_j}(\lambda_j, z_j)$ is defined in (4.4).

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