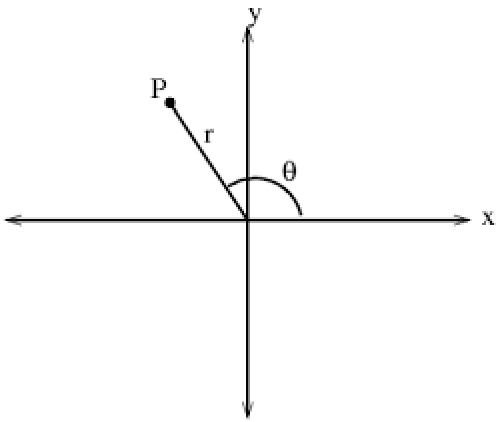
Chapter 6: Polar Coordinates and applications

Section 6.1 Polar Coordinates

Definition: The **polar coordinate system** is a two-dimensional coordinate system in which each point P on a plane is determined by a distance r from a fixed point O that is called the **pole** (or origin) and an angle θ from a fixed direction. The point P is represented by the ordered pair $(r; \theta)$ and $r; \theta$ are called **polar coordinates.**



Remark: We extend the meaning of polar coordinates $(r; \theta)$ to the case in which r is negative by agreeing that the points $(-r; \theta)$ and $(r; \theta)$ lie in the same line through O and at the same distance $|r|$ from O; but on opposite sides of O: If $r > 0$; the point $(r; \theta)$ lies in the same quadrant as θ; if $r < 0$; it lies in the quadrant on the opposite side of the pole.

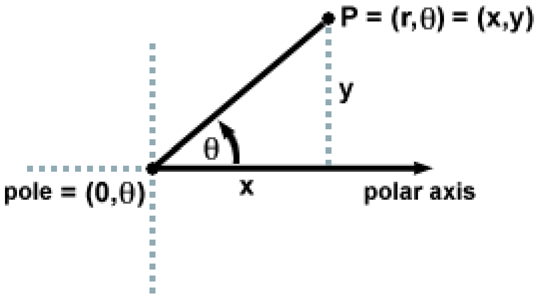
Example: Plot the points whose polar coordinates are given:

$$a)\ \left(1; \frac{5\pi}{4}\right),\quad b)\ (2; 3\pi)\quad c)\ \left(2; -\frac{2\pi}{3}\right), d)\ \left(-3; \frac{3\pi}{4}\right)$$

Remark: In the Cartesian coordinate system every point has only one representation, but in the polar coordinate system each point has many representations. For instance, the point $\left(1; \frac{5\pi}{4}\right)$ in the Example above could be written as $\left(1; -\frac{3\pi}{4}\right)$ or $\left(1; \frac{13\pi}{4}\right)$ or $\left(-1; \frac{\pi}{4}\right)$.

Relationship with Cartesian coordinates

The connection between polar and Cartesian coordinates can be seen from the figure below and described by the following formulas:



$$x = rcos\theta; y = rsin\theta \quad and$$

$$r^2 = x^2 + y^2; \ \ \theta = \tan^{-1}\frac{y}{x}; \ \ x \neq 0.$$

Example:

(a) Convert the point $(2;\frac{\pi}{3})$ from polar to Cartesian coordinates.

(b) Represent the point with Cartesian coordinates (1;-1) in terms of polar coordinates.

Section 6.2: Polar Curves

The **graph of a polar equation** $r = f(\theta)$; or more generally $F(r,\theta) = 0$; consists of all points P that have at least one polar representation $(r;\theta)$ whose coordinates satisfy the equation.

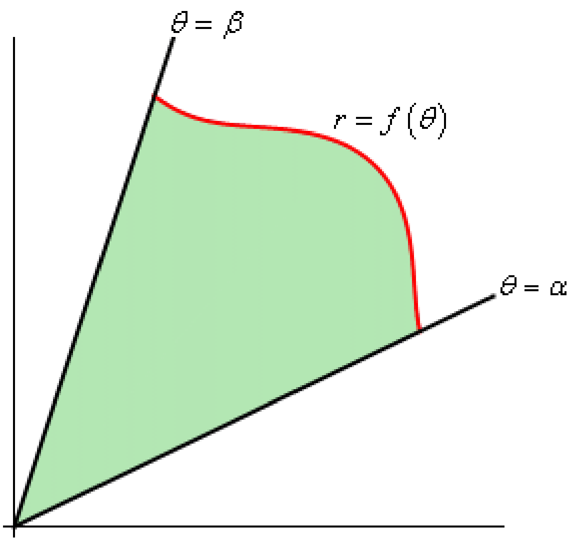
Example1: the curve $r = a$ is a circle with centre (0;0) and radius a.

Example2: Sketch the following polar curves:

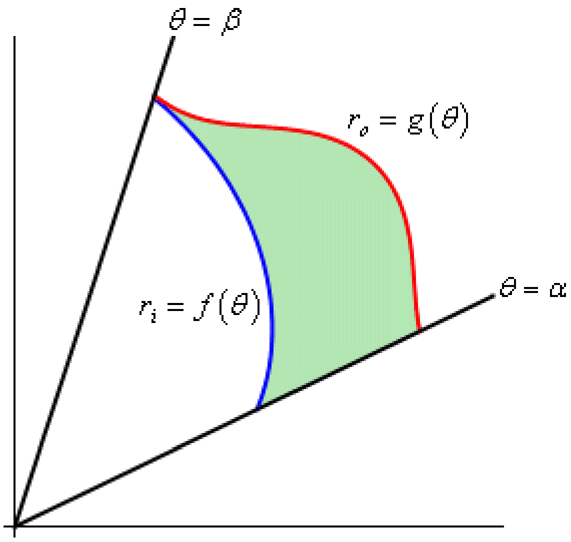
a) θ =1.
b) $r = 2, \ 0 \leq \theta \leq 2\pi$.
c) $r = 2cos\theta, \ 0 \leq \theta \leq \pi$.

Section 6.2: Area with Polar Coordinates

In this section we are going to look at areas enclosed by polar curves. These problems work a little differently in polar coordinates



We will be looking for the shaded area in the sketch above. The formula for finding this area is: $A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta = \int_{\alpha}^{\beta} \frac{1}{2} f(\theta)^2 d\theta$



$$A = \int_{\alpha}^{\beta} \frac{1}{2} [r_0{}^2 - r_1^2] d\theta = \int_{\alpha}^{\beta} \frac{1}{2} [g(\theta)^2 - f(\theta)^2] d\theta$$

Example1: Find the area of the circle with polar equation $r = 1$.
Solution:

$$A = \int_{0}^{2\pi} \frac{1}{2} r^2 d\theta = \int_{0}^{2\pi} \frac{1}{2} d\theta = \frac{1}{2} [\theta]_0^{2\pi} = \frac{1}{2} [2\pi - 0] = \pi \; u.a.$$

Example2: Find the area of region lying inside the circle with polar equation $r = 2$ and outside the circle with polar equation $r = 1$.
Solution:

$$A = \int_{\alpha}^{\beta} \frac{1}{2}[r_2{}^2 - r_1{}^2]d\theta = \int_{0}^{2\pi} \frac{1}{2}[2^2 - 1^2]d\theta = \int_{0}^{2\pi} \frac{3}{2}d\theta = \frac{3}{2}[\theta]_0^{2\pi} = \frac{3}{2}[2\pi - 0] = 3\pi\ u.a.$$

Example3: Find the area of region lying in the first quadrant and inside the circle with polar equation $r = 2$.

Solution:

$$A = \int_{\alpha}^{\beta} \frac{1}{2}[r^2]d\theta = \int_{0}^{\frac{\pi}{2}} \frac{1}{2}[2^2]d\theta = \int_{0}^{\frac{\pi}{2}} 2d\theta = 2[\theta]_0^{\frac{\pi}{2}} = 2\left[\frac{\pi}{2} - 0\right] = \pi\ u.a.$$