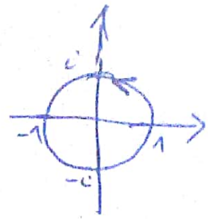
تطبيقات نظرية الرواسب لحساب بعض التكاملات.

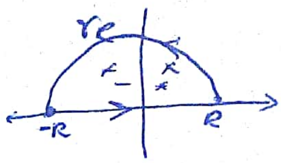
⚠



- $$\int_0^{2\pi} F(\cos t, \sin t)\, dt = \oint_{|z|=1} F\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right) \frac{dz}{iz}$$

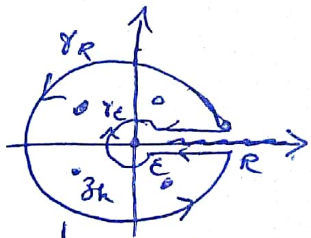
$$= 2\pi \sum_{|z_k|<1} \operatorname{Res}\left(\frac{1}{z} F\left(\left(\frac{z^2+1}{2z}\right), \left(\frac{z^2-1}{2iz}\right)\right); z_k\right)$$

- $$\mathcal{F}(f)_\omega = \hat{f}(\omega) = \int_{\mathbb{R}} f(x)\, e^{i\omega x}\, dx = 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}\left(f(z)\, e^{i\omega z}; z_k\right)$$



- $$\int_0^{+\infty} \frac{P(x)}{Q(x)}\, x^\alpha\, dx = \frac{2\pi i}{(1 - e^{2\pi i \alpha})} \sum_{z_k \in \mathbb{C} \setminus [0,\infty)} \operatorname{Res}\left(\frac{P(z)}{Q(z)} z^\alpha; z_k\right)$$

$\alpha \in \mathbb{R} \setminus \mathbb{Z}$



- $$\int_0^{+\infty} \frac{P(x)}{Q(x)} \ln x\, dx = -\frac{1}{2} \sum_{z_k \in \mathbb{C} \setminus [0,\infty)} \operatorname{Res}\left((\log z - i\pi)^2 \frac{P(z)}{Q(z)}; z_k\right)$$

$\log z = \ln|z| + i \arg z$
$0 < \arg z < 2\pi$

- $\deg Q \geq \deg P + 2$
- Q has not real roots

حساب بعض المتسلسلات

$$\sum_{n=-\infty}^{+\infty} f(n) = -\sum_{z_k \text{ أقطاب } f} \operatorname{Res}\left((\pi \cot \pi z)\, f(z); z_k\right)$$

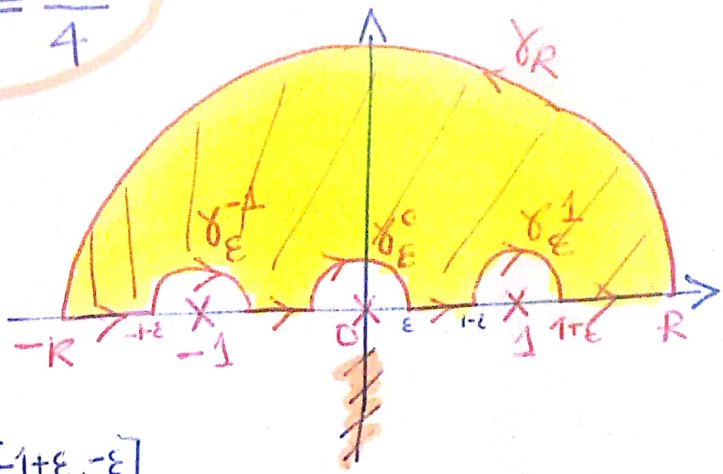
$$\sum_{n=-\infty}^{+\infty} (-1)^n f(n) = -\sum_{z_k} \operatorname{Res}\left((\pi \csc \pi z)\, f(z); z_k\right)$$

$$\sum_{n=-\infty}^{+\infty} f\left(\frac{2n+1}{2}\right) = \sum_{z_k \text{ أقطاب } f} \operatorname{Res}\left((\pi \tan \pi z)\, f(z); z_k\right)$$

$$\sum_{n=-\infty}^{+\infty} (-1)^n f\left(\frac{2n+1}{2}\right) = \sum_{z_k \text{ أقطاب } f} \operatorname{Res}\left((\pi \sec \pi z)\, f(z); z_k\right)$$

Q1] $\displaystyle \int_0^{+\infty} \frac{\ln x}{x^2-1}\,dx \overset{?}{=} \frac{\pi^2}{4}$

$f(z) = \dfrac{\log z}{z^2-1}$ نأخذ



نأخذ المسار

$$\Gamma_{R,\varepsilon} = \gamma_R \cup [-R, -1-\varepsilon] \cup \gamma_\varepsilon^{-1} \cup [-1+\varepsilon, -\varepsilon] \cup \gamma_\varepsilon^{0} \cup [\varepsilon, 1-\varepsilon] \cup \gamma_\varepsilon^{1} \cup [1+\varepsilon, R]$$

$\log z = \ln|z| + i \arg z$

$-\frac{\pi}{2} < \arg z < \pi/2$

$1 > \varepsilon > 0$

$0 < R$

f تحليلية داخل $\Gamma_{R,\varepsilon}$

فان $\displaystyle \oint_{\Gamma_{R,\varepsilon}} f(z)\,dz = 0$

$$0 = \oint_{\Gamma_{R,\varepsilon}} f(z)\,dz = \int_{\gamma_R} f(z)\,dz + \int_{-R}^{-1-\varepsilon} f(t)\,dt + \int_{\gamma_\varepsilon^{-1}} f(z)\,dz + \int_{-1+\varepsilon}^{-\varepsilon} f(t)\,dt$$
$$+ \int_{\gamma_\varepsilon^{0}} f(z)\,dz + \int_{\varepsilon}^{1-\varepsilon} f(t)\,dt + \int_{\gamma_\varepsilon^{1}} f(z)\,dz + \int_{1+\varepsilon}^{R} f(t)\,dt$$

$\displaystyle \left|\int_{\gamma_R} f(z)\,dz\right| \xrightarrow[R\to\infty]{} 0$ باستخدام مبرهنة جوردان نرى أن

- $\displaystyle \int_{-R}^{-1-\varepsilon} \frac{\log t}{t^2-1}\,dt = \int_{1+\varepsilon}^{R} \frac{\log(-t)}{t^2-1}\,dt$
- $\displaystyle \int_{-1+\varepsilon}^{-\varepsilon} \frac{\log t}{t^2-1}\,dt = \int_{\varepsilon}^{1+\varepsilon} \frac{\log(-t)}{t^2-1}\,dt$
- $\displaystyle \int_{\varepsilon}^{1-\varepsilon} \frac{\log t}{t^2-1}\,dt + \int_{1+\varepsilon}^{R} \frac{\log t}{t^2-1}\,dt \xrightarrow[R\to+\infty]{\varepsilon\to 0} \int_0^{+\infty} \frac{\log t}{t^2-1}\,dt$

II

$\circledast$ على المسار γ_ε^{-1} ; $z(\theta) = -1+\varepsilon e^{i\theta}$ حيث $0 \le \theta \le \pi$

$$\int_{\gamma_\varepsilon^{-1}} f(z)\,dz = -\int_0^{\pi} \frac{\log(-1+\varepsilon e^{i\theta})}{(-1+\varepsilon e^{i\theta})^2-1}\, i\varepsilon e^{i\theta}\,d\theta$$

$$= -\int_0^{\pi} \frac{\log(-1+\varepsilon e^{i\theta})}{\varepsilon^{2i} e^{i\theta}-2}\, i\,d\theta$$

$$= -i\int_0^{\pi} \frac{\log(-1+\varepsilon e^{i\theta})}{(-2+\varepsilon e^{i\theta})}\,d\theta \quad (*)$$

$$\lim_{z\to -1} (z+1)f(z) = \frac{\log(-1)}{-2} = -i\frac{\pi}{2}$$ لان

$$\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon^{-1}} f(z)\,dz = -\frac{\pi^2}{2} \quad (*)$$

$\circledast$ على المسار γ_ε^{0} , $z(\theta) = \varepsilon e^{i\theta}$ حيث $0 \le \theta \le \pi$

$$\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon^{0}} f(z)\,dz = \lim_{\varepsilon\to 0} -\int_0^{\pi} \frac{\log(\varepsilon e^{i\theta})}{\varepsilon^2 e^{2i\theta}-1}\, i\varepsilon e^{i\theta}\,d\theta$$

$$\left|\int_{\gamma_\varepsilon^{0}} f(z)\,dz\right| \le \int_0^{\pi} \frac{(\ln\varepsilon + i\pi)}{1-\varepsilon^2}\,\varepsilon\,d\theta \xrightarrow[\varepsilon\to 0]{} 0$$

$\circledast$ على المسار γ_ε^{1} : $z = 1+\varepsilon e^{i\theta}$

$$\int_{\gamma_\varepsilon^{1}} f(z)\,dz = -\int_0^{\pi} \frac{\log(1+\varepsilon e^{i\theta})}{(1+\varepsilon e^{i\theta})^2-1}\, i\varepsilon e^{i\theta}\,d\theta$$

$$= -i\int_0^{\pi} \frac{\log(1+\varepsilon e^{i\theta})}{\cancel{\varepsilon e^{i\theta}}(\varepsilon e^{i\theta}+2)}\,\cancel{\varepsilon e^{i\theta}}\,d\theta \xrightarrow[\varepsilon\to 0]{} 0$$

$$\lim_{z\to 1} (z-1)f(z) = \frac{\ln 1}{2} = 0$$ لان

اذن

$$0 = \int_1^{\infty} \frac{\log(-t)}{t^2-1}\,dt + \int_0^{1} \frac{\log(-t)}{t^2-1} - \frac{\pi^2}{2} + \int_0^{1} \frac{\log t}{t^2-1}\,dt + 0 + \int_1^{+\infty} \frac{\log t}{t^2-1}\,dt$$

$$0 = \int_0^{+\infty} \frac{\log t + i\pi}{t^2-1}\,dt + \int_0^{+\infty} \frac{\log t}{t^2-1}\,dt - \frac{\pi^2}{2}$$

$$I = \int_0^{+\infty} \frac{\ln t}{t^2-1}\,dt$$

$$2I + i\pi \underbrace{\int_0^{+\infty} \frac{dt}{t^2-1}}_{=0} = \frac{\pi^2}{2}$$

و بالتالي $I = \int_0^{+\infty} \frac{\ln t}{t^2-1}\,dt = \frac{\pi^2}{4}$

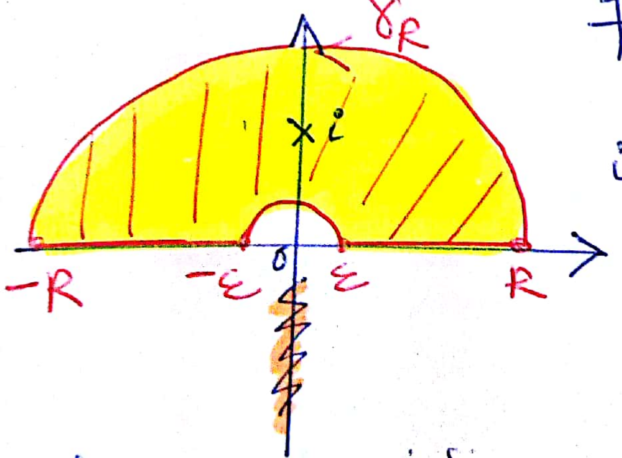
Q_2

$$J = \int_0^{+\infty} \frac{\ln x}{\sqrt{x}\,(1+x)^2}\,dx \overset{?}{=} -\frac{\pi}{4}$$

نضع $x = t^2$ فإن $dx = 2t\,dt$

$$J = \int_0^{+\infty} \frac{\ln(t^2)}{t\,(1+t^2)^2}\,2t\,dt = 4\int_0^{+\infty} \frac{\ln t}{(1+t^2)^2}\,dt$$

ندرس الدالة $f(z) = \frac{\log z}{(1+z^2)^2}$



f تحليلية على داخل $\Gamma_{R,\varepsilon}$ ماعدا i

باستخدام نظرية الرواسب

$$\oint_{\Gamma_{R,\varepsilon}} f(z)\,dz = 2\pi i\,\mathrm{Res}(f,i)$$

أين $\log z = \ln|z| + i\arg z$

حيث $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$

المسار $\Gamma_{R,\varepsilon} = \gamma_R \cup [-R,-\varepsilon] \cup \gamma_\varepsilon \cup [\varepsilon, R]$

حيث $R \gg 1$ و $0 < \varepsilon < 1$

$$\oint_{\Gamma_{R,\varepsilon}} f(z)\,dz = \int_{-R}^{-\varepsilon} f(t)\,dt + \int_{\gamma_\varepsilon} f(z)\,dz + \int_{\varepsilon}^{R} f(t)\,dt + \int_{\gamma_R} f(z)\,dz$$

$$\bullet \int_{-R}^{-\varepsilon} f(t)\,dt = \int_{\varepsilon}^{R} f(-t)\,dt = \int_{\varepsilon}^{R} \frac{\log(-t)}{(1+t^2)^2}\,dt$$

$$= \int_{\varepsilon}^{R} \frac{(\ln t + i\pi)}{(1+t^2)^2}\,dt$$

$$\bullet \int_{\varepsilon}^{R} f(t)\,dt = \int_{\varepsilon}^{R} \frac{\ln t}{(1+t^2)^2}\,dt$$

* على المسار γ_ε ، $z = \varepsilon e^{i\theta}$ حيث $0 \leq \theta \leq \pi$

$$\int_{\gamma_\varepsilon} f(z)\,dz = -\int_0^{\pi} f(\varepsilon e^{i\theta})\, i\varepsilon e^{i\theta}\,d\theta$$

$$= -\int_0^{\pi} \frac{\log(\varepsilon e^{i\theta})}{(1+\varepsilon^2 e^{2i\theta})^2}\, i\varepsilon e^{i\theta}\,d\theta$$

$$\left|\int_{\gamma_\varepsilon} f(z)\,dz\right| \leq \pi\varepsilon \frac{|\ln\varepsilon| + \pi}{(1-\varepsilon^2)^2} \xrightarrow[\varepsilon\to 0]{} 0$$

كذلك

$$\left|\int_{\gamma_R} f(z)\,dz\right| \leq \frac{R(\ln R + \pi)}{(R^2-1)^2}\,\pi \xrightarrow[R\to +\infty]{} 0$$

وبالتالي

$$\int_0^{+\infty} \frac{\ln t}{(1+t^2)^2}\,dt + \int_0^{+\infty} \frac{(\ln t + i\pi)}{(1+t^2)^2}\,dt = 2\pi i\, Res(f, i)$$

$$2\left(\int_0^{+\infty} \frac{\ln t}{(1+t^2)^2}\,dt\right) + i\pi\left(\int_0^{+\infty} \frac{dt}{(1+t^2)^2}\right) = 2\pi i\, Res(f, i)$$

$$\int_0^{+\infty} \frac{dt}{(1+t^2)^2} = i\pi\, Res\left(\frac{1}{(1+z^2)^2}, i\right) = i\pi \frac{1}{4i} = \frac{\pi}{4}$$

$$\mathrm{Res}(f,i) = \mathrm{Res}\left(\frac{\log z}{(1+z^2)^2}, i\right)$$

(i قطب من الرتبة الثانية لـ f)

$$= \frac{d}{dz}\left[\frac{\log z}{(z+i)^2}\right]_{/z=i} = \frac{4i+2\pi}{16}$$

و بالتالي

$$2\left(\int_0^{+\infty} \frac{\ln t}{(1+t^2)^2}\,dt\right) + i\pi\left(\frac{\pi}{4}\right) = 2\pi i\left(\frac{4i+2\pi}{16}\right)$$

فنجد أن

$$J = \int_0^{+\infty} \frac{\ln t}{(1+t^2)^2}\,dt = \frac{-\pi}{4}$$

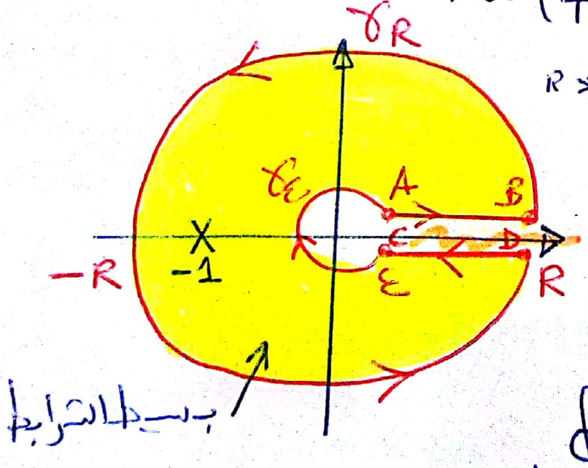
س٣

$$\int_0^{+\infty} \frac{x^{p-1}}{1+x}\,dx \overset{?}{=} \frac{\pi}{\sin(p\pi)}$$ حيث $0<p<1$

الحل: نأخذ الدالة $f(z) = \dfrac{z^{p-1}}{1+z}$ حيث $0<p<1$

$z=0$ هي نقطة فرع و $z=-1$ هو قطب بسيط لـ f.

$$\mathrm{Res}(f,-1) = \lim_{z\to -1}(z+1)f(z) = \lim_{z\to -1} z^{p-1} = e^{i(p-1)\pi}$$



نأخذ المسار $\Gamma_{R,\varepsilon}$: حيث $0<\varepsilon<1$ و $R \gg 1$

$$\Gamma_{R,\varepsilon} = [AB] \cup \gamma_R \cup [DC] \cup \gamma_\varepsilon$$

باستخدام نظرية الرواسب لدينا

$$\oint_{\Gamma_{R,\varepsilon}} f(z)\,dz = 2\pi i\, e^{i(p-1)\pi}$$

$$\oint_{R,\varepsilon} f(z)\,dz = \int_\varepsilon^R \frac{x^{p-1}}{1+x}\,dx + \int_0^{2\pi} \frac{(Re^{i\theta})^{p-1}}{1+Re^{i\theta}}\, iRe^{i\theta}d\theta + \int_R^\varepsilon \frac{(xe^{2i\pi})^{p-1}}{1+xe^{2i\pi}}\,dx + \int_{2\pi}^0 \frac{(\varepsilon e^{i\theta})^{p-1}}{1+\varepsilon e^{i\theta}}\, i\varepsilon e^{i\theta}d\theta$$

⚠ لكن وضعنا $z = xe^{2i\pi}$ بالنسبة للتكامل على [DC]
لأن زاوية z هي 2π على [DC]

باستخدام نظرية جوردان نرى

$$\left|\int_{\gamma_\varepsilon} f(z)\,dz\right| \xrightarrow[\varepsilon \to 0]{} 0$$

$$\left|\int_{\gamma_R} f(z)\,dz\right| \xrightarrow[R \to \infty]{} 0$$

و بالتالي

$$\int_0^{+\infty} \frac{x^{p-1}}{1+x}\,dx + \int_{\infty}^{0} \frac{x^{p-1} e^{2i\pi(p-1)}}{1+x}\,dx = 2\pi i\, e^{i(p-1)\pi}$$

$$\left(1 - e^{2i\pi(p-1)}\right)\left(\int_0^{+\infty} \frac{x^{p-1}}{1+x}\,dx\right) = 2\pi i\, e^{i(p-1)\pi}$$

إذاً

$$\int_0^{+\infty} \frac{x^{p-1}}{1+x}\,dx = \frac{2i\pi\, e^{i\pi(p-1)}}{1 - e^{2i\pi(p-1)}}$$

$$= \frac{2i\pi\, e^{i\pi(p-1)}}{\cancel{e^{i\pi(p-1)}}\left[e^{-i\pi(p-1)} - e^{i\pi(p-1)}\right]}$$

لما $0<p<1$

$$\int_0^{+\infty} \frac{x^{p-1}}{1+x}\,dx = \frac{\pi}{\sin(p\pi)}$$

⚠ تحويل ميلين (Mellin Transform)

$$\mathcal{M}(f)(p) = \varphi(p) = \int_0^{+\infty} x^{p-1} f(x)\,dx$$

و بالتالي

$$\mathcal{M}\left(\frac{1}{1+x}\right)(p) = \frac{\pi}{\sin(p\pi)}$$

س٤ VII

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{zt}}{\sqrt{z+1}}\,dz \overset{?}{=} \frac{e^{-t}}{\sqrt{\pi t}}$$

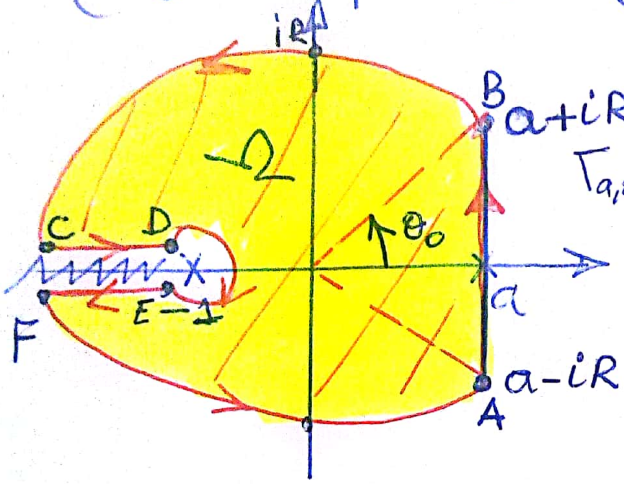
حقيقي $a>0$

حقيقي $t>0$

الحل: $\sqrt{z+1} = (z+1)^{1/2} = e^{1/2\log(1+z)}$

لدينا $z=-1$ نقطة فرع (Branch point)

نأخذ المسار $\Gamma_{a,\varepsilon,R}$ التالي:



Ω المنطقة محدودة حافتها $\Gamma_{a,\varepsilon,R}$

الدالة $f(z)=\dfrac{e^{zt}}{\sqrt{z+1}}$

تحليلية على Ω

باستخدام نظرية كوشي

$Log(1+z)=\ln|1+z|+i\,Arg(1+z)$

حيث

$-\pi < Arg(1+z) < \pi$

$$\oint_{\Gamma_{a,\varepsilon,R}} f(z)\,dz = 0$$

$$\int_{AB} f(z)dz + \int_{\widehat{BC}} f(z)dz + \int_{[CD]} f(z)dz + \int_{\widehat{DE}} f(z)dz + \int_{[EF]} f(z)dz + \int_{\widehat{FA}} f(z)dz = 0$$

$\widehat{BC}$: $z(\theta)=Re^{i\theta}$, $\theta_0 \le \theta \le \pi$

$[CD]$: $z+1=ue^{i\pi}$, $\sqrt{z+1}=i\sqrt{u}$, $z=-1-u$, $R-1<u<\varepsilon$, $-1-\varepsilon<-1-u<-R$

$\widehat{DE}$: $z=-1+\varepsilon e^{i\theta}$

$[EF]$: $z+1=ue^{-i\pi}$, $\sqrt{z+1}=-i\sqrt{u}$, $z=-1-u$, $\varepsilon<u<R-1$

$\widehat{FA}$: $z=Re^{i\theta}$, $\pi\le\theta\le 2\pi$

$$\int_{a-iR}^{a+iR}\frac{e^{zt}}{\sqrt{z+1}}dz + \underbrace{\int_{\theta_0}^{\pi}\frac{e^{Re^{i\theta}t}}{\sqrt{Re^{i\theta}+1}}iRe^{i\theta}d\theta}_{(1)} + \int_{R-1}^{\varepsilon}\frac{e^{-(u+1)t}}{i\sqrt{u}}(-du) + \underbrace{\int_{\pi}^{-\pi}\frac{e^{(\varepsilon e^{i\theta}-1)t}}{\sqrt{\varepsilon e^{i\theta}+1}}i\varepsilon e^{i\theta}d\theta}_{(2)}$$

$$+\underbrace{\int_{\pi}^{2\pi-\theta_0}\frac{e^{Re^{i\theta}t}}{\sqrt{Re^{i\theta}+1}}iRe^{i\theta}d\theta}_{(3)} = 0$$

التكاملات (1) و (2) و (3) تساوي صفر إذا كانت $\varepsilon\to 0$, $R\to\infty$

$$\int_{a-i\infty}^{a+i\infty} \frac{e^{zt}}{\sqrt{z+1}}\,dz - \int_{+\infty}^{0} \frac{e^{-(u+1)t}}{i\sqrt{u}}\,du = 0$$

$$\int_{a-i\infty}^{a+i\infty} \frac{e^{zt}}{\sqrt{z+1}}\,dz = -\int_{0}^{+\infty} \frac{e^{-(u+1)t}}{i\sqrt{u}}\,du$$

نضع $u = v^2$ فإن $du = 2v\,dv$

$$\int_0^{\infty} \frac{e^{-ut}e^{-t}}{i\sqrt{u}}\,du = \frac{2e^{-t}}{i}\int_0^{\infty} \frac{e^{-v^2 t}}{\cancel{v}}\cancel{v}\,dv$$

$w = v\sqrt{t}$

$dw = dv\sqrt{t}$

$\frac{dw}{\sqrt{t}} = dv$

$$= -2i\frac{e^{-t}}{\sqrt{t}}\left(\int_0^{+\infty} e^{-w^2}\,dw\right) = -\cancel{2}i\,\frac{e^{-t}}{\sqrt{t}}\,\frac{\sqrt{\pi}}{\cancel{2}}$$

وبالتالي

$t>0$

$a>0$

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{zt}}{\sqrt{z+1}}\,dz = \frac{e^{-t}}{\sqrt{\pi t}}$$

$$I = \int_0^{+\infty} e^{-w^2}\,dw = ?$$

$$I^2 = \left(\int_0^{+\infty} e^{-w^2}\,dw\right)\left(\int_0^{+\infty} e^{-u^2}\,du\right) = \iint_{0\ 0}^{+\infty\ +\infty} e^{-(u^2+w^2)}\,du\,dw$$

$$= \int_0^{\pi/2}\left(\int_0^{\infty} e^{-r^2}\, r\,dr\right)d\theta = \left(\int_0^{\pi/2} d\theta\right)\left(-\frac{1}{2}\left(\int_0^{\infty} -2r\,e^{-r^2}\,dr\right)\right)$$

$$= \frac{\pi}{2}\left(-\frac{1}{2}\left[e^{-r^2}\right]_0^{\infty}\right) = \frac{\pi}{4}$$

وبالتالي $I = \int_0^{+\infty} e^{-w^2}\,du = \frac{\sqrt{\pi}}{2}$.

تحويل ميلين العكسي (Inverse Mellin transform)

$$\mathcal{M}^{-1}(\varphi)(x) = f(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} x^{-s}\varphi(s)\,ds$$

$$\mathcal{M}(f)(s) = \varphi(s) = \int_0^{+\infty} x^{s-1} f(x)\,dx.$$

$$= \int_0^{+\infty} x^{s} f(x)\,\underbrace{\frac{dx}{x}}_{\text{Haar measure}}$$