جامعة الملك سعود
King Saud University

College of Computer & Information Sciences

Muzahmiyah branch

Math106 (Integral Calculus)

Name: Answer

ID:

Time: 2 hours | *Final Exam* | Maximum Marks: 40

1. Evaluate the following integrals:

- $\int \frac{\sqrt{x}}{\sqrt{1+x^3}} dx$ [3]

1 put $u = x^{3/2}$; $du = \frac{3}{2} x^{1/2} dx$ so $\sqrt{x}\, dx = \frac{2}{3} du$

$I = \int \frac{\sqrt{x}}{\sqrt{1+x^3}} dx = \frac{2}{3} \int \frac{du}{\sqrt{1+u^2}}$ 1 $= \frac{2}{3} \sinh^{-1}(u) + c$

$= \frac{2}{3} \sinh^{-1}(x^{3/2}) + c$, $c \in \mathbb{R}$. 1

- $\int x e^x dx$ Integration by parts [3]

1 $u(x) = x \Rightarrow u'(x) = 1$

$v'(x) = e^x \quad v(x) = e^x$

$\int x e^x dx =$ 1 $x e^x - \int e^x dx = x e^x - e^x + c$ 1

$= (x-1) e^x + c$, $c \in \mathbb{R}$

- $\int \cos^3 x dx = \int \cos^2 x \cos x \, dx$ [3]

$= \int (1 - \sin^2 x) \cos x \, dx$ 1

$u = \sin x$

$du = \cos x \, dx$ So $\int \cos^3 x \, dx = \int (1-u^2) du$ 1

$= u - \frac{u^3}{3} + c$

$= \sin x - \frac{\sin^3 x}{3} + c$ 1

- $\int \tan^2 x dx$ $= \int (\sec^2 x - 1)\, dx$ 2 [3]

$= \tan x - x + c, \quad c \in \mathbb{R}$ 1

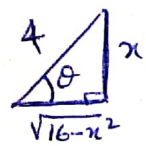
- $\int \frac{dx}{x^2\sqrt{16-x^2}}$ (trigonometric substitution)

1 put $x = 4\sin\theta$ then $dx = 4\cos\theta\, d\theta$ [3]

$\sqrt{16-x^2} = \sqrt{16-4^2\sin^2\theta} = \sqrt{16(1-\sin^2\theta)} = \sqrt{16\cos^2\theta} = 4\cos\theta$

$I = \int \frac{dx}{x^2\sqrt{16-x^2}} = \int \frac{\cancel{4}\cancel{\cos\theta}\, d\theta}{16\sin^2\theta\, \cancel{4}\cancel{\cos\theta}} = \frac{1}{16}\int \frac{1}{\sin^2\theta}\, d\theta = \frac{1}{16}\int \csc^2\theta\, d\theta$ 1

$= -\frac{1}{16}\cot\theta + c = -\frac{1}{16}\frac{\sqrt{16-x^2}}{x} + cst.$ 1



$x = 4\sin\theta$

$\sin\theta = \frac{x}{4}$

- $\int \frac{x}{x^2-4x+8}dx$ by complete square

$\underbrace{x^2-4x+8}$ $= \underbrace{(x-2)^2 - 4 + 8} = (x-2)^2 + 2^2$ [3]

So $\int \frac{x}{x^2-4x+8}\, dx = \int \frac{x}{2^2+(x-2)^2}\, dx = \frac{1}{2}\int \frac{2x-4+4}{x^2-4x+8}\, dx$

$= \frac{1}{2}\int \frac{(2x-4)}{x^2-4x+8}\, dx + 2\int \frac{dx}{x^2-4x+8}$ 1

$= \frac{1}{2}\ln(x^2-4x+8) + 2\int \frac{dx}{2^2+(x-2)^2}$ 1

$= \frac{1}{2}\ln(x^2-4x+8) + \tan^{-1}\left(\frac{x-2}{2}\right) + cst.$ 1

- $\int \frac{dx}{(x-1)(x+2)}$

$\int \frac{dx}{(x-1)(x+2)} = \int \left[\frac{A}{x-1} + \frac{B}{x+2}\right] dx$ [3] 1

$= A\ln|x-1| + B\ln|x+2| + cst$ 1

put $f(x) = \frac{1}{(x-1)(x+2)}$

1
- $A = \lim_{x\to 1} (x-1) f(x) = \lim_{x\to 1} \frac{1}{x+2} = \frac{1}{3}$
- $B = \lim_{x\to -2} (x+2) f(x) = \lim_{x\to -2} \frac{1}{x-1} = -\frac{1}{3}$

So $\int \frac{dx}{(x-1)(x+2)}\, dx = \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + cst.$

2. i) Find the following limit: $\lim_{x\to 0^+} (\tan x)^{\tan x}$. $= 0^0$ (I.F), $\tan 0 = 0$

[3]

1 $\ln\left[(\tan x)^{\tan x}\right] = (\tan x)\ln(\tan x)$

- $\lim_{x\to 0^+} \ln\left[(\tan x)^{\tan x}\right] = \lim_{x\to 0^+} (\tan x)\ln(\tan x) = 0.\infty$

$$= \lim_{x\to 0^+} \frac{\ln(\tan x)}{\left(\frac{1}{\tan x}\right)} = \frac{\infty}{\infty}$$

by Hospital's rule 1 $$= \lim_{x\to 0^+} \frac{\frac{\sec^2 x}{\tan x}}{-\csc^2 x} = -\lim_{x\to 0^+} \tan x = 0$$

1 So $\lim_{x\to 0^+} (\tan x)^{\tan x} = 1$

ii) Show that the integral $\int_1^{+\infty} \frac{\ln x}{x^2}dx$ converges and find its value.

[3]

0,5 • $\int_1^{+\infty} \frac{\ln x}{x^2}dx = \lim_{t\to+\infty}\left(\int_1^t \frac{\ln x}{x^2}dx\right)$

$$\int_1^t \frac{\ln x}{x^2}dx = -\left[\frac{1}{x}\ln x\right]_1^t + \int \frac{1}{x^2}dx = -\left[\frac{1}{x}\ln x\right]_1^t - \left[\frac{1}{x}\right]_1^t$$

1,5 $\ln 1 = 0$

$u(x) = \ln x \quad u'(x) = 1/x$

$v'(x) = \frac{1}{x^2} \Rightarrow v(x) = -\frac{1}{x}$

1 So $\int_1^{+\infty} \frac{\ln x}{x^2}dx = \lim_{t\to+\infty}\left[-\frac{1}{t}\ln t - \frac{1}{t} + 1\right] = 1$ because $\lim_{t\to+\infty} \frac{\ln t}{t} = 0$

3. (a) Sketch the region R bounded by the curves $y = 2 - x^2$, $y = x$ and find its area.

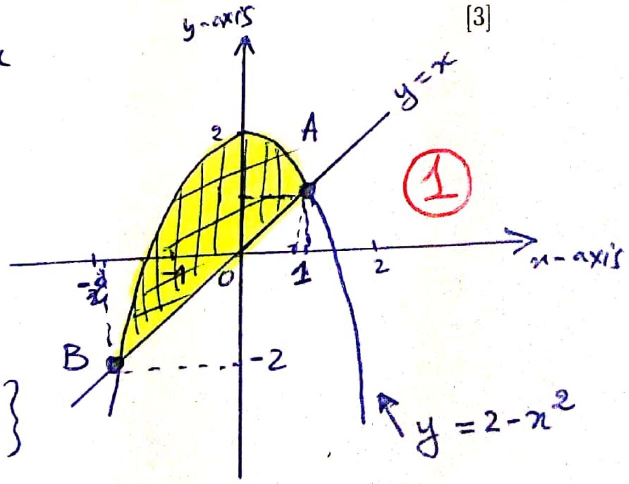
[3]

- Intersection points: $y = 2 - x^2 = x$

$x^2 + x - 2 = 0$

$(x-1)(x+2) = 0$

1 $A\begin{cases} x = 1 \\ y = 1 \end{cases}$ or $B\begin{cases} x = -2 \\ y = -2 \end{cases}$



- The region R is

$$R = \left\{(x,y) \;/\; \begin{array}{l} -2 \le x \le 1 \\ x \le y \le 2-x^2 \end{array}\right\}$$

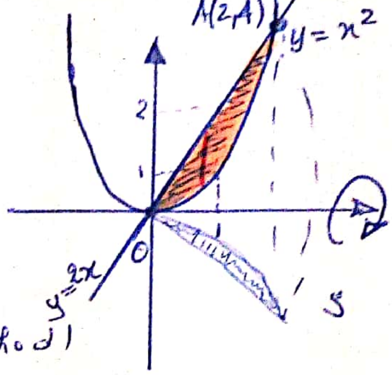
- The area of R is

1 $$A(R) = \int_{-2}^{1}\left[(2-x^2) - x\right]dx = \left[2x - \frac{x^3}{3} - \frac{x^2}{2}\right]_{-2}^{1}$$

$$A(R) = \left(2 - \frac{1}{3} - \frac{1}{2}\right) - \left(-4 + \frac{8}{3} - 2\right)$$

$$A(R) = 4 - 3 - \frac{1}{2} + 4 = 9/2$$

(b) Find the volume of the solid generated by revolving the region bounded by the curves: $y = x^2$ and $y = 2x$, about the x-axis. [3]



• Intersection points :

$y = x^2 = 2x$

$x^2 - 2x = 0 \Leftrightarrow x(x-2) = 0$

$x = 0$ or $x = 2$

1 • The region is

$R = \left\{ (x,y) \,/\, \begin{array}{l} 0 \le x \le 2 \\ x^2 \le y \le 2x \end{array} \right\}$

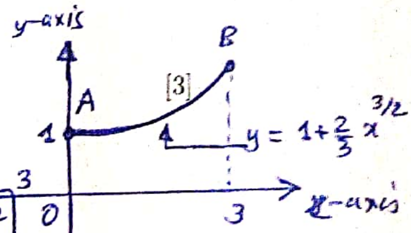
Using washer method (Disk method)

• The volume of the solid S is given by

1 $V(S) = \pi \int_0^2 \left[(2x)^2 - (x^2)^2\right] dx = \pi \int_0^2 (4x^2 - x^4)\, dx$

1 $V(S) = \pi \left[4 \, x^3/3 - x^5/5 \right]_0^2 = 2^5 \pi \left[1/3 - 1/5 \right] = \frac{64\pi}{15}.$

(c) Determine the length of the curve $y = 1 + \frac{2}{3}x^{3/2}$ from $x = 0$ to $x = 3$. [3]



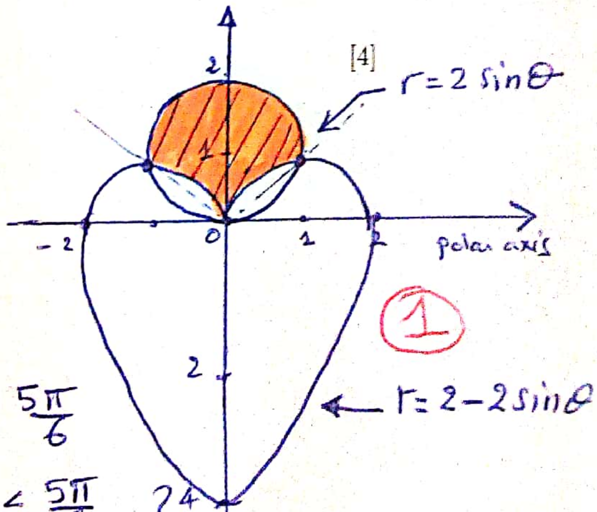
1 • $\ell(\widehat{AB}) = \int_0^3 \sqrt{1 + (f'(x))^2}\, dx$

$f(x) = 1 + \frac{2}{3}x^{3/2}$; $f'(x) = x^{1/2} = \sqrt{x}$

So $\ell(\widehat{AB}) = \int_0^3 \sqrt{1+x}\, dx = \int_0^3 (1+x)^{1/2} dx = \frac{2}{3}\left[(1+x)^{3/2}\right]_0^3$ 1

$= 2/3\,[8-1] = 14/3$ 1

(d) Sketch the region R that lies inside the curve $r = 2\sin\theta$ and outside the curve $r = 2 - 2\sin\theta$, and find its area. [4]



• $r = 2\sin\theta$: is an eq of a circle

• $r = 2 - 2\sin\theta$: is an eq of a cardioid

Intersection points

1 $r = 2\sin\theta = 2 - 2\sin\theta$

$\sin\theta = 1 - \sin\theta$

$2\sin\theta = 1 \Rightarrow \sin\theta = \frac{1}{2}$

So $\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

The region R is

1 $R = \left\{ (r,\theta) \,/\, \begin{array}{l} \frac{\pi}{6} \le \theta \le \frac{5\pi}{6} \\ 2-2\sin\theta \le r \le 2\sin\theta \end{array} \right\}$

The area is given by

1 $A(R) = \frac{1}{2} \int_{\pi/6}^{5\pi/6} \left[(2\sin\theta)^2 - (2-2\sin\theta)^2 \right] d\theta.$

by symmetry $A(R) = \int_{\pi/6}^{\pi/2} \left[4\sin^2\theta - 4(1 + \sin^2\theta - 2\sin\theta) \right] d\theta$

$= 4\int_{\pi/6}^{\pi/2} \left[2\sin\theta - 1 \right] d\theta = 4\left[2\cos\theta - \theta \right]_{\pi/6}^{\pi/2}$

$A(R) = 4(\sqrt{3} - \pi/3).$