HomeWork n°2 (Math 106) (Dr Bothen)

Ex 1 : Find the limit if there exists

$$\lim_{x \to 5} \left[\frac{1}{x-5} - \frac{1}{\ln(x-4)} \right]$$

Ex 2 : Evaluate the following Integrals:

$$I = \int x^4 \ln x \, dx$$

$$J = \int \sin^2 x \cos^5 x \, dx$$

$$K = \int \frac{x^2}{\sqrt{16 - x^2}} \, dx$$

$$L = \int \frac{5x - 1}{x^2 + 2x + 10} \, dx$$

$$M = \int \frac{dx}{\sqrt{-x^2 + 4x}}$$

$$N = \int \frac{dx}{6 + 3\sin x}$$

Ex3 Determine the following improper integral

$$\int_0^{+\infty} x^2 e^{-3x^3} \, dx$$ converges or diverges.

Ex4 Sketch the region bounded by $x = y^2 + 2y + 4$ and $x = 6y + 4$ and find its area.

Answer sheet HW2 (Math 106)

Ex1

- $$\lim_{x\to 5}\left[\frac{1}{x-5}-\frac{1}{\ln(x-4)}\right]=\infty-\infty \text{ IF}$$

$$=\lim_{x\to 5}\left[\frac{\ln(x-4)-(x-5)}{(x-5)\ln(x-4)}\right]=\frac{0}{0}$$

Hospital's Rule

$$=\lim_{x\to 5}\frac{\frac{1}{x-4}-1}{\ln(x-4)+(x-5)\frac{1}{x-4}}=\lim_{x\to 5}\frac{1-(x-4)}{(x-4)\ln(x-4)+(x-5)}$$

$$=\lim_{x\to 5}\frac{5-x}{(x-4)\ln(x-4)+(x-5)}=\lim_{x\to 5}\frac{-1}{\ln(x-4)+(x-4)\frac{1}{(x-4)}+1}=-\frac{1}{2}$$

Ex2

- $$I=\int x^4\ln x\,dx \qquad \begin{array}{l} u'(x)=x^4\\ V(x)=\ln x\end{array} \Rightarrow \begin{array}{l} U(x)=\frac{x^5}{5}\\ V'(x)=\frac{1}{x}\end{array}$$

$$=\frac{x^5}{5}\ln x-\frac{1}{5}\int\frac{x^5}{x}dx$$

$$I=\frac{x^5}{5}\ln x-\frac{1}{25}x^5+cst$$

- $$J=\int \sin^2 x\cos^5 x\,dx$$

$$=\int \sin^2 x\cos^4 x\cos x\,dx$$

$$=\int \sin^2 x(1-\sin^2 x)^2\cos x\,dx \quad ; \quad u=\sin x$$

$$=\int u^2(1-u^2)^2du=\int u^2(u^4-2u^2+1)\,du$$

$$=\int[u^6-2u^4+u^2]\,du=\frac{u^7}{7}-\frac{2u^5}{5}+\frac{u^3}{3}+cst$$

$$=\frac{\sin^7 x}{7}-\frac{2\sin^5 x}{5}+\frac{\sin^3 x}{3}+cst.$$

- $$K=\int\frac{x^2}{\sqrt{16-x^2}}dx=\int\frac{x^2}{\sqrt{4^2-x^2}}dx=\int\frac{16\sin^2\theta\,\cancel{4\cos\theta}\,d\theta}{\cancel{4\cos\theta}}$$

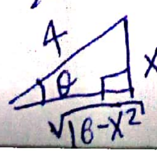
put $x=4\sin\theta$

$dx=4\cos\theta\,d\theta$

$\sqrt{16-x^2}=4\cos\theta$

As $x=4\sin\theta$ then

$\sin\theta=\frac{x}{4}$ so $\theta=\sin^{-1}\left(\frac{x}{4}\right)$



$\cos\theta=\frac{\sqrt{16-x^2}}{4}$

$$K=16\int\sin^2\theta\,d\theta$$

$$=16\int\frac{1-\cos 2\theta}{2}d\theta$$

$$=8\left[\theta-\frac{\sin(2\theta)}{2}\right]+c$$

$$=8\theta-8\sin\theta\cos\theta+c$$

$$=8\sin^{-1}\left(\frac{x}{4}\right)-\frac{1}{2}x\sqrt{16-x^2}+c$$

- $L = \int \frac{5x-1}{x^2+2x+10}\,dx = \int \frac{5x-1}{(x+1)^2-1+10}\,dx$

$$= \frac{5}{2}\int \frac{2x+2}{x^2+2x+10}\,dx - 6\int \frac{dx}{3^2+(x+1)^2}$$

$$= \frac{5}{2}\ln(x^2+2x+10) - \frac{6}{3}\tan^{-1}\left(\frac{x+1}{3}\right)+C$$

$$L = \frac{5}{2}\ln(x^2+2x+10) - 2\tan^{-1}\left(\frac{x+1}{3}\right)+C$$

- $M = \int \frac{dx}{\sqrt{-x^2+4x}}$

$$-x^2+4x = -[x^2-4x] = -[(x-2)^2-2^2] = 2^2-(x-2)^2$$

$$M = \int \frac{dx}{\sqrt{2^2-(x-2)^2}} = \sin^{-1}\left(\frac{x-2}{2}\right)+cst.$$

- $N = \int \frac{dx}{6+3\sin x}$ we put $u = \tan(x/2)$

$$dx = \frac{2du}{1+u^2}$$

$$\sin x = \frac{2u}{1+u^2}$$

$$N = \int \frac{\frac{2du}{1+u^2}}{6+3\left(\frac{2u}{1+u^2}\right)}$$

$$N = 2\int \frac{du}{6(1+u^2)+6u} = \frac{1}{3}\int \frac{du}{u^2+u+1} = \frac{1}{3}\int \frac{du}{(u+1/2)^2-\frac{1}{4}+1}$$

$$N = \frac{1}{3}\int \frac{du}{\left(\frac{\sqrt{3}}{2}\right)^2+\left(u+\frac{1}{2}\right)^2} = \frac{1}{3}\,\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2u+1}{\sqrt{3}}\right)+cst$$

$$= \frac{2}{3\sqrt{3}}\tan^{-1}\left(\frac{2\tan(x/2)+1}{\sqrt{3}}\right)+cst.$$

Ex3

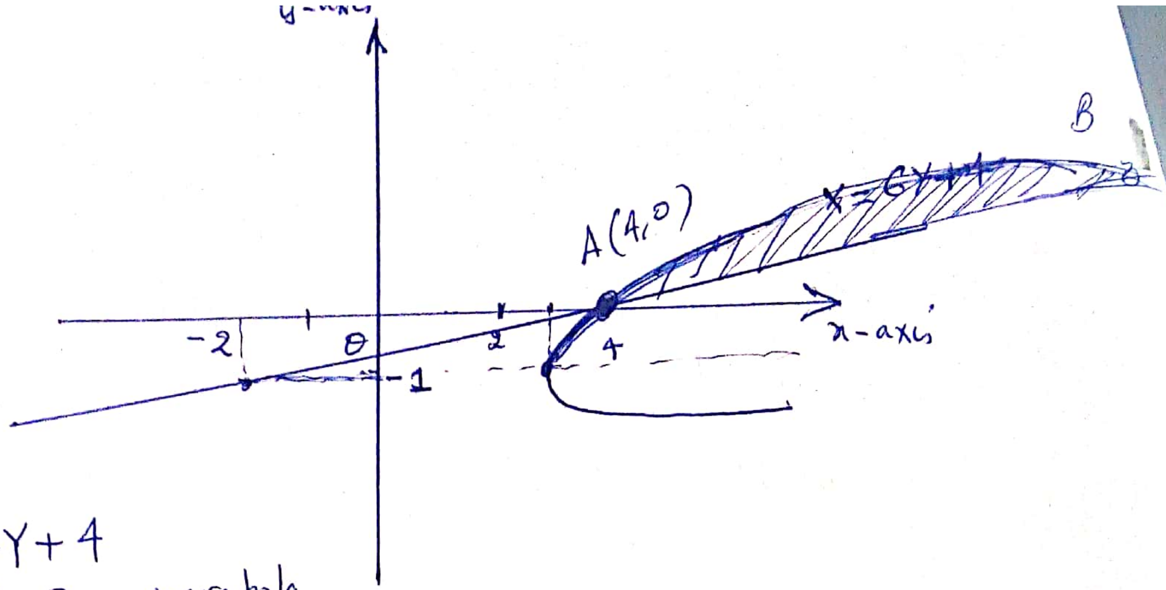
$$\int_0^{+\infty} x^2 e^{-3x^3}\,dx = \lim_{t\to+\infty}\left(\int_0^t x^2 e^{-3x^3}\,dx\right)$$

$$\int_0^t x^2 e^{-3x^3}\,dx = -\frac{1}{9}\int_0^t (-9x^2)\, e^{-3x^3}\,dx = -\frac{1}{9}\left[e^{-3x^3}\right]_0^t$$

$$= -\frac{1}{9}\left[e^{-3t^3}-1\right]$$

As $\lim_{t\to+\infty} e^{-3t^3} = 0$ then $\int_0^{+\infty} x^2 e^{-3x^3}\,dx$ converges to $\frac{1}{9}$.

Ex4



$X = Y^2 + 2Y + 4$

$X = (Y+1)^2 + 3$ parabola

- Intersection points $X = 6Y + \cancel{4} = Y^2 + 2Y + \cancel{4}$

$$Y^2 - 4Y = 0$$
$$y(y-4) = 0 \quad \text{so} \quad \begin{cases} y = 0 \\ x = 4 \end{cases} \quad \text{or} \quad \begin{cases} y = 4 \\ X = 28 \end{cases}$$

- Our region is $R = \left\{ (x,y) \,/\, \begin{array}{l} 0 \le y \le 4 \\ y^2+2y+4 \le x \le 6y+4 \end{array} \right\}$

Its area is $A(R) = \int_0^4 \left[(6y+4) - (y^2+2y+4) \right] dy$

$$= \int_0^4 \left[-y^2 + 4y \right] dy$$

$$= \left[-y^3/3 + 2y^2 \right]_0^4 = -\frac{64}{3} + 32$$

$$= 32 \left[1 - 2/3 \right] = \frac{32}{3} > 0 .$$