Home Work 2 (Math 106) Sec 1054

Ex1 Evaluate the following integrals

$$I = \int x^4 \ln x \, dx$$

$$J = \int \tan^{-1} x \, dx$$

$$K = \int \sin^2 x \cos^5 x \, dx$$

$$L = \int \frac{x^2}{\sqrt{9-x^2}} \, dx$$

$$M = \int \frac{x^2+3}{(x-1)^2(x+1)} \, dx$$

$$N = \int \frac{dx}{\sqrt{x^2+2x}}$$

$$P = \int \frac{dx}{x^{1/2}+x^{1/3}}$$

Ex 2: Determine the following improper integral $\displaystyle\int_0^{+\infty} \frac{x}{(x^2+1)^2} \, dx$ converges or diverges.

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Answer Sheet HW2 (Math 106)

Ex1

- $I = \int x^4 \ln x \, dx = \frac{1}{5} x^5 \ln x - \frac{1}{5}\int \frac{1}{x} x^5 dx$

$u(x) = \ln x \qquad \Rightarrow \qquad u'(x) = \frac{1}{x}$

$v'(x) = x^4 \qquad\qquad\quad v(x) = \frac{x^5}{5}$

$$\boxed{I = \int x^4 \ln x \, dx = \frac{1}{5} x^5 \ln x - \frac{1}{25} x^5 + cst}$$

- $J = \int \tan^{-1} x \, dx = x \tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2} dx$

$u(x) = \tan^{-1} x \qquad \Rightarrow \qquad u'(x) = \frac{1}{1+x^2}$

$v'(x) = 1 \qquad\qquad\quad v(x) = x$

So $\boxed{J = \int \tan^{-1} x \, dx = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + cst}$

- $K = \int \sin^2 x \cos^5 x \, dx = \int \sin^2 x \, (\cos^2 x)^2 \cos x \, dx$

$= \int \sin^2 x \, (1 - \sin^2 x)^2 \cos x \, dx \underset{u = \sin x}{=} \int u^2 (1-u^2)^2 du$

$= \int u^2 (u^4 - 2u^2 + 1) du = \int (u^6 - 2u^4 + u^2) du$

$= \frac{u^7}{7} - 2\frac{u^5}{5} + \frac{u^3}{3} + cst$

$$\boxed{K = \int \sin^2 x \cos^5 x \, dx = \frac{\sin^7 x}{7} - \frac{2 \sin^5 x}{5} + \frac{\sin^3 x}{3} + cst}$$

- $L = \int \frac{x^2}{\sqrt{9-x^2}} dx$

Trigonometric Substitution

put $x = 3 \sin\theta$

$dx = 3 \cos\theta \, d\theta$

$9 - x^2 = 9 - 9\sin^2\theta = 9\cos^2\theta$

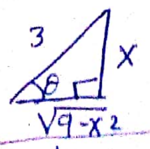
$\sqrt{9-x^2} = 3\cos\theta$

$L = \int \frac{9 \sin^2\theta \, \cancel{3}\cancel{\cos\theta} \, d\theta}{\cancel{3}\cancel{\cos\theta}}$

$= 9 \int \sin^2\theta \, d\theta = 9 \int \left(\frac{1 - \cos 2\theta}{2}\right) d\theta = \frac{9}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] + c$

As $x = 3\sin\theta$ then $\sin\theta = x/3$

so $\theta = \sin^{-1}(x/3)$



As $\sin 2\theta = 2 \sin\theta\cos\theta = 2(x/3)\frac{\sqrt{9-x^2}}{3}$

we get $\boxed{L = \int \frac{x^2}{\sqrt{9-x^2}} dx = \frac{9}{2}\sin^{-1}(x/3) - \frac{1}{2} x\sqrt{9-x^2} + cst}$

• $M = \int \frac{x^2+3}{(x-1)^2(x+1)}\,dx = \int\left[\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}\right]dx$

$f(x) = \frac{x^2+3}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$

$B = \lim_{x\to 1}(x-1)^2 f(x) = \lim_{x\to 1} \frac{x^2+3}{x+1} = \frac{4}{2} = \boxed{2 = B}$

$C = \lim_{x\to -1}(x+1)\,f(x) = \lim_{x\to -1} \frac{x^2+3}{(x-1)^2} = \frac{4}{4} = \boxed{1 = C}$

$f(0) = \frac{3}{1} = -A + B + C = -A + 2 + 1$ so $\boxed{A = 0}$

$M = \int \frac{x^2+3}{(x-1)^2(x+1)}\,dx = 2\int \frac{dx}{(x-1)^2} + \int \frac{dx}{x+1} = \frac{-2}{x-1} + \ln|1+x| + cst$

• $N = \int \frac{dx}{\sqrt{x^2+2x}} = \int \frac{dx}{\sqrt{(x+1)^2-1^2}} = \cosh^{-1}(x+1) + cst.$

$x^2+2x = (x+1)^2 - 1$ (complete square).

Δ $\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}(u/a) + c$

$u > a.$

• $P = \int \frac{dx}{x^{1/2}+x^{1/3}} = 6\int \frac{u^5\,du}{u^3+u^2} = 6\int \frac{u^5}{u^2(u+1)}\,du$

$= 6\int \frac{u^3}{u+1}\,du$

put $u = x^{1/n}$ with $n = lcm(2,3) = 6$

$u = x^{1/6} \Leftrightarrow u^6 = x$. So $dx = 6u^5\,du$

$u^3 = \sqrt{x}$

$u^2 = x^{1/3}$

$$\begin{array}{r|l} u^3 & u+1 \\ \hline -u^3-u^2 & u^2-u+1 \\ -u^2 & \\ +u^2+u & \\ u & \\ -u-1 & \\ -1 & \end{array}$$

So $\frac{u^3}{u+1} = (u^2-u+1) - \frac{1}{1+u}$ for $u \neq -1$

$P = \int \frac{dx}{x^{1/2}+x^{1/3}} = 6\int \frac{u^3}{u+1}\,du = 6\int\left[u^2-u+1 - \frac{1}{1+u}\right]du$

$= 6\left[\frac{u^3}{3} - \frac{u^2}{2} + u - \ln|1+u|\right] + cst = 2\sqrt{x} - 3x^{1/3} + 6x^{1/6}$

$- 6\ln|1+x^{1/6}| + cst.$

Ex2

$$\int_0^{+\infty} \frac{x}{(x^2+1)^2}\,dx := \lim_{t\to+\infty} \left(\int_0^t \frac{x}{(x^2+1)^2}\,dx \right)$$

$$\frac{1}{2}\int_0^t 2x\,(x^2+1)^{-2}\,dx = -\frac{1}{2}\left[(x^2+1)^{-1} \right]_0^t = -\frac{1}{2}\left[\frac{1}{1+t^2} - 1 \right]$$

Then $$\int_0^{+\infty} \frac{x}{(x^2+1)^2}\,dx = -\frac{1}{2} \lim_{t\to+\infty} \left[\frac{1}{1+t^2} - 1 \right]$$

$$= \frac{1}{2}$$

So the improper integral converges to $1/2$.