Solution of M 107 Final Examination (First Semester 1439-1440)

Question:1. (a) Solve system of linear equations by using Gaussian Elimination method

[6+6+6]

$$3x - y + 6z = 6$$
$$x + y + z = 2.$$
$$2x + y + 4z = 3$$

(b) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 5 \end{bmatrix}$ **Find, if possible, a matrix B such that AB = C.**

(c) Given the matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 3 \\ 3 & 1 & -2 \end{bmatrix}$**, find adj A, and verify that**

$$A(adj\ A) = |A|I$$

Solution

ⓐ $[A|b] = \left[\begin{array}{ccc|c} 3 & -1 & 6 & 6 \\ 1 & 1 & 1 & 2 \\ 2 & 1 & 4 & 3 \end{array}\right] \equiv \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 3 & -1 & 6 & 6 \\ 2 & 1 & 4 & 3 \end{array}\right] \equiv \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -4 & 3 & 0 \\ 0 & 0 & -5 & 4 \end{array}\right]$

$\equiv \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -\frac{3}{4} & 0 \\ 0 & 0 & 1 & -\frac{4}{5} \end{array}\right].$ $\boxed{3}$

$x + y + z = 2$
$y - \frac{3}{4}z = 0 \quad \Rightarrow$
$z = -\frac{4}{5}$

$z = -\frac{4}{5}$
$y = \frac{3}{4}z = \frac{3}{4}\left(-\frac{4}{5}\right) = -\frac{3}{5}$
$x = -y - z + 2 = \frac{17}{5}$ $\boxed{3}$

solution is $x = 17/5$, $y = -3/5$, $z = -4/5$.

ⓑ $AB = C$, $A^{-1}AB = A^{-1}C$, $B = A^{-1}C$

$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 3 & 2 & 0 & 0 & 1 \end{array}\right] \equiv \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -3 & 2 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 2 & 3 & -2 \end{array}\right] = [I|A^{-1}]$ $\boxed{3}$

$B = A^{-1}C = \begin{bmatrix} -1 & -3 & 2 \\ -1 & -1 & 1 \\ 2 & 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -10 \\ -1 & 1 & 5 \\ 3 & 0 & -10 \end{bmatrix}.$ $\boxed{3}$

ⓒ $A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 3 \\ 3 & 1 & -2 \end{bmatrix}$ $\quad C = \begin{bmatrix} -3 & 7 & -1 \\ 8 & -14 & 5 \\ 6 & -7 & 2 \end{bmatrix}$

$adj\ A = \begin{bmatrix} -3 & 8 & 6 \\ 7 & -14 & -7 \\ -1 & 5 & 2 \end{bmatrix}$, $\det A = 7$ $\boxed{1}$

$\boxed{3}$

$A(adj A) = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 3 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} -3 & 8 & 6 \\ 7 & -14 & -7 \\ -1 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

$= 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|I.$

$\boxed{2}$

Question:2.(a) Find sides and angles of triangle whose vertices are
[6+6]

$$A\ (1,-2,2),\ B\ (2,1,-1)\ \text{and}\ C(3,-1,2)$$

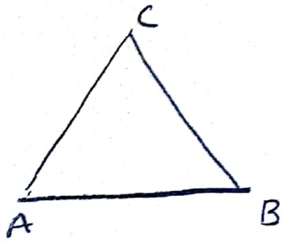
(b) Determine whether the lines intersect, and if so find the point of intersection

$$x = 1 + 3t,\ y = 1 + 5t,\ z = -4 + 2t, \text{and}$$

$$x = 1 + 3v,\ y = 4 + 2v,\ z = -1 - v,\quad v, t \in R.$$

Solution

(a)



$$\overrightarrow{AB} = \langle 1, 3, -3 \rangle$$

$$\overrightarrow{AC} = \langle 2, 1, 0 \rangle$$

$$\overrightarrow{BC} = \langle 1, 2, 3 \rangle$$

Sides $\|AB\| = \sqrt{19}$, $\|AC\| = \sqrt{5}$, $\|BC\| = \sqrt{19}$ $\boxed{2}$

$$\cos A = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\|\overrightarrow{AB}\| \|\overrightarrow{AC}\|} = \frac{2+3}{\sqrt{19}\sqrt{5}} = \sqrt{\frac{5}{19}}, \qquad A = \cos^{-1}\sqrt{\frac{5}{19}}$$

$$\cos B = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\|BA\| \|BC\|} = \frac{-1-6+9}{\sqrt{19}\sqrt{14}} = \frac{2}{\sqrt{19}\sqrt{14}}, \qquad B = \cos^{-1}\sqrt{\frac{2}{(19)(7)}}$$

$$\cos C = \frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{\|\overrightarrow{CA}\| \cdot \|\overrightarrow{CB}\|} = \frac{2+2+0}{\sqrt{5}\sqrt{14}} = \frac{4}{\sqrt{5}\sqrt{14}}, \quad C = \cos^{-1}\left(\frac{4}{\sqrt{5}\sqrt{14}}\right) \qquad \boxed{4}$$

(b) Let the point of intersection be (x_0, y_0, z_0)

$$\begin{aligned} 1 + 3t_0 &= 1 + 3v_0 \\ 1 + 5t_0 &= 4 + 2v_0 \\ -4 + 2t_0 &= -1 - v_0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 3t_0 - 3v_0 &= 0 & \to E_1 \\ 5t_0 - 2v_0 &= 3 & \to E_2 \\ 2t_0 + v_0 &= 3 & \to E_3 \end{aligned}$$

From E_1 and E_2 $\quad t_0 = 1,\ v_0 = 1.$ $\boxed{4}$

t_0, v_0 satisfy E_3 $\Rightarrow$ Lines intersect

Point of intersection. $(4, 6, -2)$ $\boxed{2}$

Question:3. (a) Show that $\lim_{(x,y)\to(0,0)}\left(\frac{(x+y)^2}{x^2+y^2}\right)$ does not exist.

[6+6+6]

(b) Let $f(x,y,z)=\sqrt{x^2+y^2+z^2}$, use differentials to approximate the change in $f(x,y,z)$, when the point (x,y,z) moves from (4,3,0) to (4.2, 2.9, 0.2).

(c) The position vector of a moving particle at time t is given by

$$r(t)=2e^t i+3e^{-t} j+2\sqrt{3}\,t\,k.$$

Find at t = 0, the tangential and the normal components of acceleration.

Solution

(a) Along $x=0$ $\lim_{(0,y)\to(0,0)} \frac{y^2}{y^2} = \lim_{(0,y)\to(0,0)} 1 = 1 \rightarrow L_1$ $\boxed{2}$

Along $y=x$ $\lim_{(x,x)\to(0,0)} \frac{(x+x)^2}{x^2+x^2} = \lim_{(x,x)\to(0,0)} \frac{4x^2}{2x^2} = \lim_{(x,x)\to(0,0)} 2 = 2 \rightarrow L_2$ $\boxed{2}$

$L_1 \neq L_2 \Rightarrow \lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}$ does not exist. $\boxed{2}$

(b)

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}}dx + \frac{y}{\sqrt{x^2+y^2+z^2}}dy + \frac{z}{\sqrt{x^2+y^2+z^2}}dz \quad \boxed{3}$$

$x=4,\ y=3,\ z=0,\ dx=.2,\ dy=-.1,\ dz=.2$ $\boxed{1}$

$$df = \frac{4}{5}(.2) + \frac{3}{5}(-.1) + 0 = \frac{.8}{5} - \frac{.3}{5} = \frac{.5}{5} = .1 \quad \boxed{2}$$

(c)

$r(t) = \langle 2e^t, 3e^{-t}, 2\sqrt{3}t\rangle$

$r'(t) = \langle 2e^t, -3e^{-t}, 2\sqrt{3}\rangle$ $\boxed{1}$

$r''(t) = \langle 2e^t, 3e^{-t}, 0\rangle$

At $t=0$, $r'(0) = \langle 2, -3, 2\sqrt{3}\rangle$

$r''(0) = \langle 2, 3, 0\rangle$ $\boxed{1}$

$r'(0)\cdot r''(0) = 4-9+0 = -5$

$$r'(0)\times r''(0) = \begin{vmatrix} i & j & k \\ 2 & -3 & 2\sqrt{3} \\ 2 & 3 & 0 \end{vmatrix} = \langle -6\sqrt{3}, 4\sqrt{3}, 12\rangle \quad \boxed{1}$$

$\|r'(0)\| = \sqrt{4+9+12} = \sqrt{25} = 5$

$\|r'(0)\times r''(0)\| = \sqrt{108+48+144} = \sqrt{300} = 10\sqrt{3}$ $\boxed{1}$

Tangential component of accel. $a_T = \frac{r'\cdot r''}{\|r'\|} = \frac{-5}{5} = -1$ $\boxed{1}$

Normal component of accel $a_N = \frac{\|r'\times r''\|}{\|r'\|} = \frac{10\sqrt{3}}{5} = 2\sqrt{3}$. $\boxed{1}$

Question:4. (a) Show that $f(x,y)=e^{x}(x\cos y-y\sin y)$ satisfies the Laplace equation

[6+6+8]

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0.$$

(b) Find the angle between the tangent planes to the surfaces $x\log z=y^2-1$, and $x^2y=2-z$ at the point $(1,1,1)$

(c) Let $f(x,y)=3x^3-2xy^2+5y$, and the vector $\vec{u}=<\frac{4}{5},\frac{3}{5}>$.

(i) Find the directional derivative of $f(x,y)$ at $P(1,2)$ in the direction of $\vec{u}$.

(ii) Find equation of tangent plane to surface $z=f(x,y)$ at the point $(1,2,5)$.

(iii) Find equation of normal line to surface $z=f(x,y)$ at the point $(1,2,5)$.

Solution.

(a)

$f_x = e^x[x\cos y - y\sin y] + e^x[\cos y]$

$\boxed{2}$ $f_{xx} = e^x[x\cos y - y\sin y] + e^x[\cos y] + e^x[\cos y]$

$f_y = e^x[-x\sin y - y\cos y - \sin y]$

$\boxed{2}$ $f_{yy} = e^x[-x\cos y + y\sin y - \cos y - \cos y]$

$f_{xx}+f_{yy} = e^x[x\cos y - y\sin y + 2\cos y] + e^x[-(x\cos y - y\sin y + 2\cos y)]$

$\boxed{2}$ $= 0$.

(b) $f(x,y,z) = x\log z - y^2 + 1 = 0$, $g(x,y,z) = x^2y + z - 2 = 0$

$\nabla f(x,y,z) = \langle \log z, -2y, \frac{x}{z}\rangle$, $\nabla g(x,y,z) = \langle 2xy, x^2, 1\rangle$ $\boxed{2}$

$N_1 = \nabla f\langle 1,1,1\rangle = \langle 0,-2,1\rangle$

$N_2 = \nabla g(1,1,1) = \langle 2,1,1\rangle$ $\boxed{2}$

$\cos\theta = \dfrac{N_1\cdot N_2}{\|N_1\|\|N_2\|} = \dfrac{-1}{\sqrt{5}\sqrt{6}} \Rightarrow \theta = \cos^{-1}\left(\dfrac{-1}{\sqrt{30}}\right)$ $\boxed{2}$

(c) $z = f(x,y) = 3x^3 - 2xy^2 + 5y$

$F(x,y,z) = 3x^3 - 2xy^2 + 5y - z$

$\nabla F = \langle 9x^2 - 2y^2, -4xy + 5, -1\rangle$

$N = \nabla F(1,2,5) = \langle 1,-3,-1\rangle$ $\boxed{2}$

(i) $\nabla f|_{(1,2)} = \langle 1,-3\rangle$

$D_u f(1,2) = \langle 1,-3\rangle\cdot\langle \frac{4}{5},\frac{3}{5}\rangle$

$= \frac{4}{5} - \frac{9}{5} = -\frac{5}{5} = -1$ //

$\boxed{2}$

(ii) Equation of Tangent plane at $(1,2,5)$

$(x-1) - 3(y-2) - (z-5) = 0$

$x - 3y - z + 10 = 0$ $\boxed{2}$

(iii) Equation of Normal line at $(1,2,5)$

$x = 1+t,\ y = 2-3t,\ z = 5-t,\quad t\in\mathbb{R}.$

$\boxed{2}$

Question:5. (a) Find local extrema and saddle points of $f(x,y) = x^4 + y^3 + 32x - 9y$.

[6+6] **(b)** If $f(x,y) = 4x^2 - 4xy + y^2$, use Lagrange multipliers to find the extrma of $f(x,y)$ subject to condition $x^2 + y^2 = 1$.

Solution

(a) $f_x = 4x^3 + 32$, $f_{xx} = 12x^2$

$f_y = 3y^2 - 9$, $f_{xy} = 0$

$f_{yy} = 6y$

- critical points $4x^3 + 32 = 0$, $3y^2 - 9 = 0$ $\Rightarrow$ $x = -2$, $y = \pm\sqrt{3}$ $\boxed{2}$

$(-2, \sqrt{3})$, $(-2, -\sqrt{3})$

- Discriminant $D(x,y) = f_{xx} f_{yy} - f_{xy}^2 = 72x^2 y$ $\boxed{2}$

At $(-2, -\sqrt{3})$, $D(-2, -\sqrt{3}) = -288\sqrt{3} < 0$ $\Rightarrow$ There is saddle point $(-2, -\sqrt{3}, f(-2, -\sqrt{3}))$ $\boxed{2}$

At $(-2, \sqrt{3})$ $D(-2, \sqrt{3}) = 288\sqrt{3} > 0$ $\Rightarrow$ Local extrema $\boxed{2}$

$f_{xx}(-2, \sqrt{3}) = 48 > 0$ $\Rightarrow$ Local Minima.

(b) $f(x,y) = y^2 - 4xy + 4x^2$ $g(x,y) = x^2 + y^2 - 1 = 0$

$\nabla f = \langle -4y + 8x, 2y - 4x \rangle$, $\nabla g = \langle 2x, 2y \rangle$ $\boxed{2}$

$\nabla f = \lambda \nabla g$

$\langle -4y + 8x, 2y - 4x \rangle = \lambda \langle 2x, 2y \rangle$

$-4y + 8x = 2\lambda x$ $\rightarrow E_1$ $E_1 + 2E_2$

$2y - 4x = 2\lambda y$ $\rightarrow E_2$ $2\lambda x + 4\lambda y = 0$

$x^2 + y^2 = 1$ $\rightarrow E_3$ $2\lambda(x + 2y) = 0$

$\Rightarrow$ either $\lambda = 0$, or $x + 2y = 0$

$x = -2y$

$E_3 \Rightarrow x^2 + 4x^2 = 1$, $5x^2 = 1$, $x^2 = \frac{1}{5}$, $x = \pm\frac{1}{\sqrt{5}}$, $y = \mp\frac{2}{\sqrt{5}}$ $\boxed{2}$

Points are

$\left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$, $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$, $\left(\frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)$, $\left(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$

$f(x,y)$ $\frac{13}{5}$ 0 0 $\frac{13}{5}$

Local Max at $\left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$, $\left(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ $\boxed{2}$

Local Min at $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$, $\left(\frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)$.