King Saud University
College of sciences
Department of mathematics
Second semester 1431/1432 H

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Time: 2 hours

Second Midterm Exam Math 580 (Theory Measure I)

Exercise 1:(7 points)

1. Let $(X, \mathcal{A})$ be a measurable space and (μ_j) be a sequence of positives measures on $\mathcal{A}$. Assume that $\forall A \in \mathcal{A}$ and $\forall j \in \mathbb{N}$, $\mu_j(A) \leq \mu_{j+1}(A)$. For all $A \in \mathcal{A}$, we put $\mu(A) = \sup_{j \in \mathbb{N}} \mu_j(A)$.
   Show that μ is a measure on $\mathcal{A}$.

2. On the measurable space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, we define for all $j \in \mathbb{N}$,
$$\nu_j(A) = \text{card}(A \cap [j, +\infty]).$$
   Show that ν_j is a measure on $\mathcal{P}(\mathbb{N})$ and $\nu_j(A) \geq \nu_{j+1}(A)$.

3. Let $\nu : \mathcal{P}(\mathbb{N}) \longrightarrow [0, \infty]$ defined by $\nu(A) = \inf_{j \in \mathbb{N}} \nu_j(A)$. Compute $\nu(\mathbb{N})$ and $\nu(\{k\})$ for $k \in \mathbb{N}$. Deduce that ν is not a measure.

Exercise 2: (6 points)

Let $(X, \mathcal{A}, \mu)$ a probability space. Let $\phi : \mathbb{R} \longrightarrow \mathbb{R}^+$ be twice differentiable function with $\phi'' \geq 0$ (ie ϕ is convex) and let $f : (X, \mathcal{A}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a measurable function such that $\int_X f(x)d\mu(x) < \infty$.

1. Show that for all $z, y \in I$ (I is an interval) with $z \leq y$,
$$\phi(y) \geq \phi(z) + \phi'(z)(y - z).$$

2. Deduce $\phi\left(\int_X f(x)d\mu(x)\right) \leq \int_X (\phi \circ f)(x)d\mu(x)$. *(Jensen's inequality)*

3. Prove that for all $f : [0, 1] \longrightarrow \mathbb{R}$ such that $\int_0^1 f(x)d\mu(x) < \infty$,
$$\left(\int_0^1 |f(x)|d\mu(x)\right)^2 \leq \int_0^1 (f(x))^2 d\mu(x).$$

Exercise 3:(7 points)

1. Compute for $a < 0$, $a = 0$ and $a > 0$,

$$\lim_{n\to+\infty} \int_a^{+\infty} \frac{n+\sin x}{1+n^2x^2}dx.$$

2. Compute for all $k \in \mathbb{N}$,

$$\lim_{n\to+\infty} \int_0^{n} x^k \left(1-\frac{x}{n}\right)^n dx.$$

<u>*Hint:*</u> *Remember that:* $\int_0^{+\infty} x^k e^{-x} dx = k!$ *and* $1 - z \leq e^{-z}$, $\forall z \in \mathbb{R}$.

3. Compute $\lim\limits_{n\to+\infty} \sum\limits_{k=0}^{n} (-1)^k \left(\int_0^{1-\frac{1}{n}} \frac{x^k}{k!} dx \right)$.

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Solution Second Midterm Exam Math 580 (Theory Measure I)

Exercise n°I (7 points)

① - we have $\mu(\phi) = \sup_{j\in\mathbb{N}} \mu_j(\phi) = 0$

- Let $(A_n)_n$ be a disjoint sequence of elements of $\mathcal{X}$ (pairwise)

we put $A = \bigcup_{n\geq 1} A_n$

For all j and n, we have $\mu_j(A) \geq \mu_j(A_1) + \dots + \mu_j(A_n)$

We take the limit of both sides for $j \to \infty$

$$\mu(A) \geq \mu(A_1) + \dots + \mu(A_n)$$

By limit when $n \to \infty$,

$$\mu(A) \geq \sum_{n\geq 1} \mu(A_n) \quad (1)$$

. If $\sum_{n\geq 1} \mu(A_n) = \infty$ then $\mu(A) = \infty$

(2) . we suppose that $\sum_{n\geq 1} \mu(A_n) < \infty$. Then $\forall \varepsilon > 0$, $\exists i$ such that

$\sum_{n>i} \mu(A_i) \leq \varepsilon$. So $\forall j$, $\mu_j(A) \leq \mu_j(A_1) + \dots + \mu_j(A_i) + \varepsilon$ (*)

because μ_j is a measure

When $j \to \infty$, (*) becomes $\mu(A) \leq \mu(A_1) + \dots + \mu(A_i) + \varepsilon$

As ε is arbitrary, we get $\mu(A) \leq \sum_{n\geq 1} \mu(A_n)$ (2)

From (1) and (2), we deduce:

$$\mu(A) = \sum_{n\geq 1} \mu(A_n)$$

②

$$\nu_j : \mathcal{P}(\mathbb{N}) \longrightarrow [0,\infty]$$
$$A \longmapsto \nu_j(A) = \text{card}(A \cap [j,\infty])$$

. $\nu_j(\phi) = \text{card}(\{\phi\}) = 0$

. Let $(A_n)_n$ be a disjoint pairwise sequence of subsets of $\mathbb{N}$

(2)

$$\nu_j\left(\bigcup_{n\geq 1} A_n\right) = \text{card}\left(\bigcup_{n\geq 1} A_n \cap [j,\infty]\right)$$

$$= \text{card} \bigcup_{n\geq 1} (A_n \cap [j,\infty])$$

$$= \sum_{n\geq 1} \text{card}(A_n \cap [j,\infty]) \quad \text{because } (A_n) \text{ is disjoint sequence}$$

$$= \sum_{n\geq 1} \nu_j(A_n)$$

$\forall j$, ν_j is a measure on $\mathcal{P}(\mathbb{N})$.

As $A \cap [j,\infty] \supseteq A \cap [j+1,\infty]$, we obtain $\nu_j(A) \geq \nu_{j+1}(A)$

③ $\nu : \mathcal{P}(\mathbb{N}) \longrightarrow [0,\infty]$

$$A \longrightarrow \nu(A) := \inf_{j\in\mathbb{N}} \nu_j(A)$$

$\forall j$, $\mathbb{N} \cap [j,\infty]$ is infinite card $(\mathbb{N} \cap [j,\infty]) = \infty$, So $\nu(\mathbb{N}) = \infty$

Let $k \in \mathbb{N}$, there exists $j \in \mathbb{N}$ such that $j > k$, Then $\{k\} \cap [j,\infty] = \emptyset$

we deduce $\nu(\{k\}) = 0$.

If ν is a measure then $\underbrace{\nu(\mathbb{N})}_{\substack{\| \\ \infty}} = \sum_{k\in\mathbb{N}} \nu(\{k\}) = 0$ ↯ contradiction

So ν is not a measure on $\mathcal{P}(\mathbb{N})$.

Exercise n°2 (6 points)

① By fundamental Theorem of Calculus, For $z \leq y$

$$\varphi(y) - \varphi(z) = \int_z^y \varphi'(t)\,dt$$

As $\varphi'' \geq 0$ then φ' is increasing Then $\varphi'(t) \geq \varphi'(z)$ $\forall t \geq z$

We get

$$\varphi(y) - \varphi(z) \geq \int_z^y \varphi'(z)\,dt$$

$$\varphi(y) - \varphi(z) \geq \varphi'(z)(y-z) \Leftrightarrow \varphi(y) \geq \varphi(z) + \varphi'(z)(y-z)$$

②) We put $z = \int_X f(x)\,d\mu(x)$ and $y = f(x)$

by ①, we get: $\varphi(f(x)) \geq \varphi\left(\int_X f(x)\,d\mu(x)\right) + \varphi'\left(\int_X f(x)\,d\mu(x)\right)\cdot\left(f(x) - \int_X f(x)\,d\mu(x)\right)$

$$(\varphi\circ f)(x) \geq \varphi\left(\int_X f(x)\,d\mu(x)\right) + \varphi'\left(\int_X f(x)\,d\mu(x)\right)\left(f(x) - \int_X f(x)\,d\mu(x)\right)$$

By integrating,

$$\int_X (\varphi\circ f)(x)\,d\mu(x) \geq \varphi\left(\int_X f(x)\,d\mu(x)\right)\underbrace{\int_X d\mu(x)}_{\substack{\| \\ 1}} + \varphi'\left(\int_X f(x)\,d\mu(x)\right)\int_X\left(f(x) - \int_X f(x)\,d\mu(x)\right)$$

$$\geq \varphi\left(\int_X f(x)\,d\mu(x)\right) + \varphi'\left(\int_X f(x)\,d\mu(x)\right)\underbrace{\left(\int_X f(x)\,d\mu(x) - \int_X f(x)\,d\mu\right)}_{=0}$$

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It follows that: $\int_X (\varphi \circ f)(x)\, d\mu(x) \underset{\text{Jensen's inequality}}{\geq} \varphi\left(\int_X f(x)\, d\mu(x)\right)$

(3) We take $d\mu$ the Lebesgue measure on $[0,1]$,

$$\mu([0,1]) = \int_{[0,1]} d\mu(x) = 1 \quad \text{and} \quad \varphi(t) = t^2$$

(φ is convex because $\varphi'' = 2 > 0$)

(2) If $f : [0,1] \to \mathbb{R}$ measurable and $\int_0^1 f(x)\, d\mu(x) < \infty$

then by Jensen's inequality, we obtain:

$$\left(\int_0^1 |f(x)|\, d\mu(x)\right)^2 \leq \int_0^1 (f(x))^2\, d\mu(x).$$

⚠ We can get this inequality directely by applying Hölder inequality $g \equiv 1$, $p=2$, $q=2$

Exercise n° 3 (7 points)

(1) Let $n \geq 1$. Integration by substitution $y = nx$ then $dy = n\,dx$

$$\int_a^\infty \frac{n + \sin x}{1 + n^2 x^2}\, dx = \int_{na}^\infty \frac{1 + \frac{\sin(y/n)}{n}}{1+y^2}\, dy$$

The function $f(x) = \frac{1}{1+x^2}$ is continuous and positive on $\mathbb{R}$ and

we know $\int_{\mathbb{R}} \frac{dx}{1+x^2} = \pi < \infty$

In other hand, we have $\left| \frac{1 + 1/n \cdot \sin(y/n)}{1+y^2} \right| \leq \frac{2}{1+y^2}$.

So By Dominate Convergence Thm,

(1) • For $a > 0$,

$$\lim_{n\to\infty} \int_{na}^{+\infty} \frac{1 + 1/n\, \sin(y/n)}{1+y^2}\, dy = \lim_{n\to\infty} \int_{\mathbb{R}} \chi_{[na,\infty)}(y) \frac{1 + 1/n\, \sin(y/n)}{1+y^2}\, dy$$
$$= 0.$$

• For $a < 0$,

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(1) $$\lim_{n\to\infty}\int_{na}^{+\infty}\frac{1+\frac{1}{n}\sin(y/n)}{1+y^2}\,dy=\lim_{n\to\infty}\int_{\mathbb{R}}\chi_{[na,+\infty)}(y)\,\frac{1+\frac{1}{n}\sin(y/n)}{1+y^2}\,dy$$

$$=\int_{\mathbb{R}}\frac{dy}{1+y^2}=\pi$$

- For $a=0$,

(1) $$\int_{na}^{+\infty}\frac{1+\frac{1}{n}\sin(\frac{y}{n})}{1+y^2}\,dy=\int_0^{+\infty}\frac{1+\frac{1}{n}\sin(y/n)}{1+y^2}\,dy$$

$$\lim_{n\to\infty}\int_0^{\infty}\frac{1+\frac{1}{n}\sin(\frac{y}{n})}{1+y^2}\,dy=\int_0^{\infty}\frac{dy}{1+y^2}=\frac{\pi}{2}.$$

(2) For $n\geq 1$, Let $f_n(x)=\chi_{[0,n]}(x)\cdot x^k\cdot(1-\frac{x}{n})^n$, $x\geq 0$

it is the product of Borelian functions, Then f_n is a Borelian function.

As $1-z\leq e^{-z}$ $\forall z\in\mathbb{R}$ and $n\geq 1$, it follows that

$$0\leq f_n(x)\leq x^k e^{-x}\qquad\forall x\geq 0$$

(2) Moreover

$$\lim_{n\to\infty}f_n(x)=x^k e^{-x},\ \forall x\geq 0\quad\text{and}$$

$\int_0^{\infty}x^k e^{-x}\,dx=k!<\infty$, Then by Dominate convergence

Thm, $$\lim_{n\to\infty}\int_0^n x^k\left(1-\frac{x}{n}\right)^n dx=\int_0^{\infty}x^k e^{-x}\,dx=k!$$

(3) Let $u_n=\chi_{[0,1-1/n]}(x)\sum_{k=0}^{n}\frac{(-1)^k}{k!}x^k$, $\forall\ 0\leq x<1$

$$|u_n(x)|\leq\chi_{[0,1-1/n]}(x)\sum_{k=0}^{n}\frac{x^k}{k!}\leq e^x$$

(2) $$\lim_{n\to\infty}u_n=e^{-x}\qquad\forall x\in[0,1[$$

By Dominate convergence Thm, we get

$$\lim_{n\to\infty}\sum_{k=0}^{n}(-1)^k\left(\int_0^{1-1/n}\frac{x^k}{k!}\,dx\right)=\lim_{n\to\infty}\int_0^1 u_n\,dx=\int_0^1\lim u_n\,dx=\int_0^1 e^{-x}dx$$

$$=1-e^{-1}.$$

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